Classification

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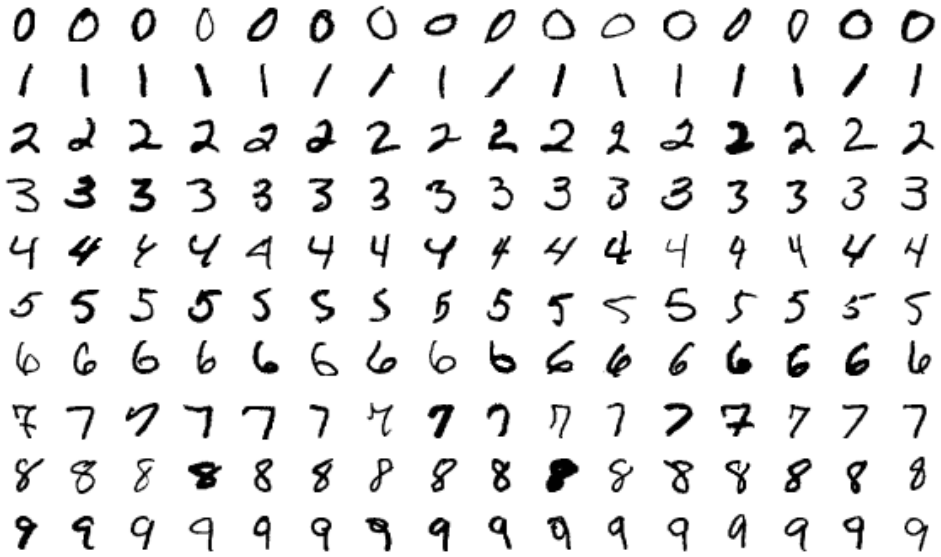
Classification problem

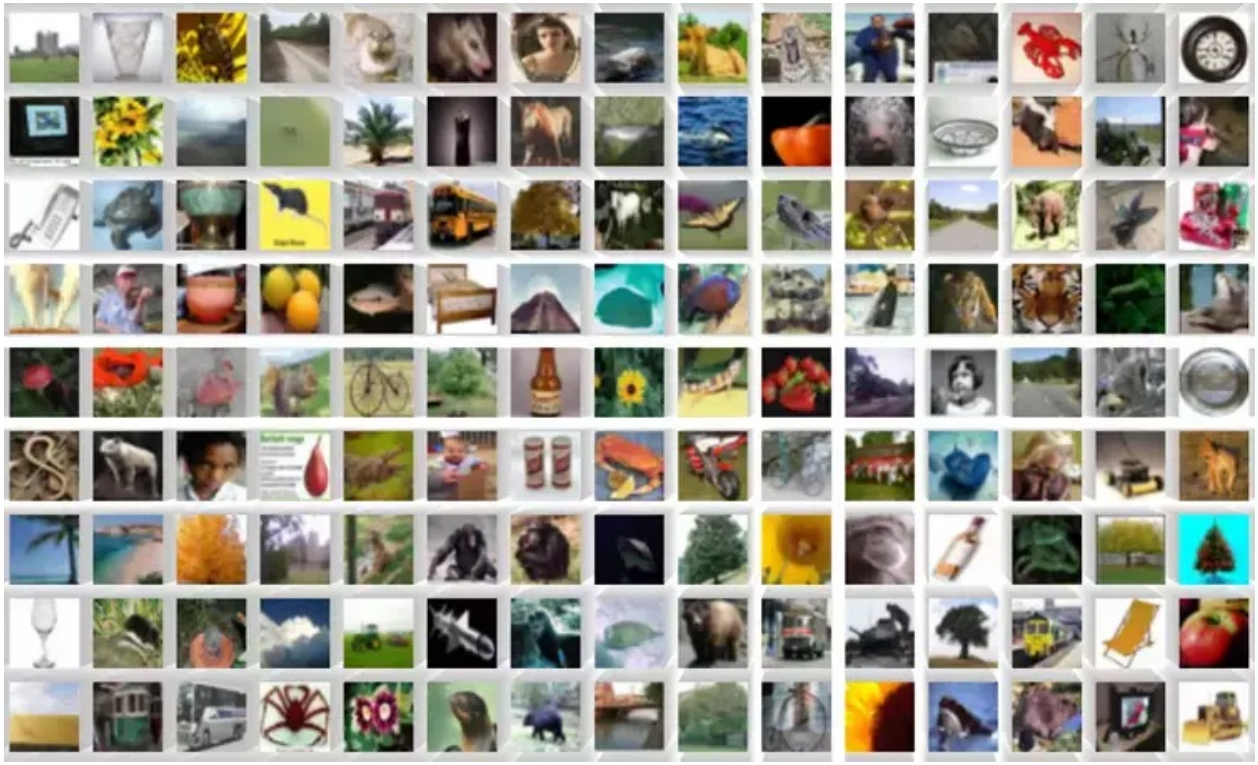
- Classification: assign a label (or category, class) to an observation based on its features
- $\mathcal{X}$: input space (e.g. $\mathbb{R}^d$); $\mathcal{Y}$: output space (e.g. $\{1, 2, \ldots, K\}$)
- $x \in \mathcal{X}$: feature vector, input, data point...
- $y \in \mathcal{Y}$: label, category, class...
- Classifier: a mapping $f : \mathcal{X} \to \mathcal{Y}$
- Goal: construct a classifier f that accurately predicts the label y given the features x

MNIST dataset



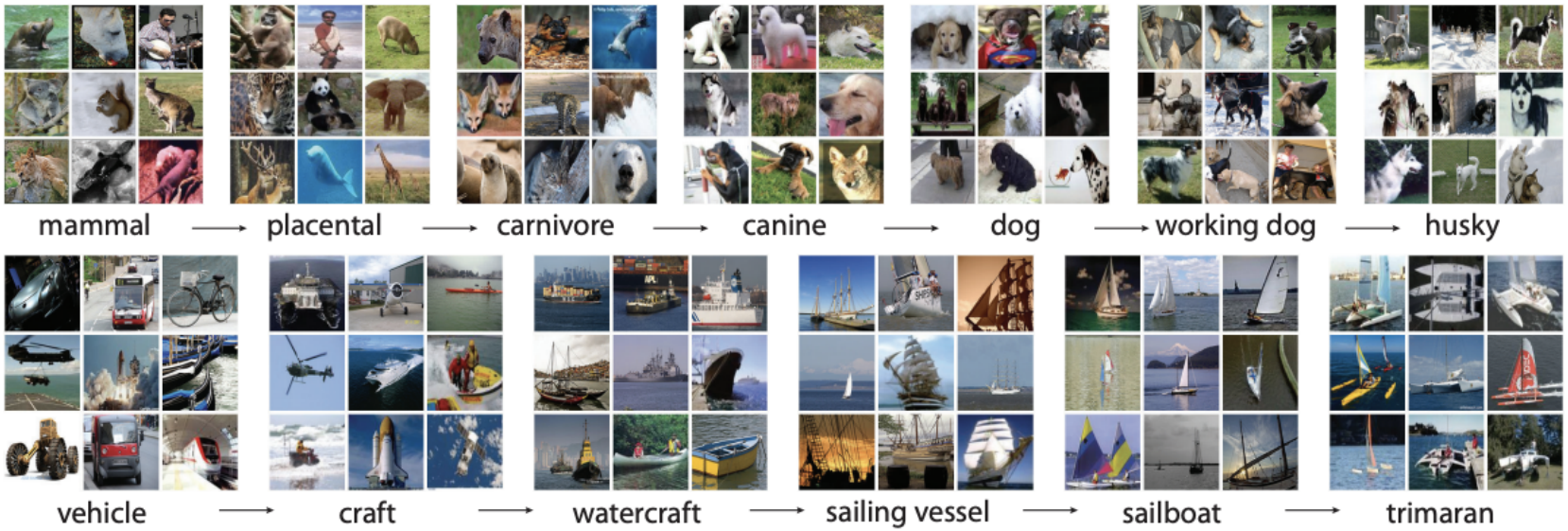
- Input: 28x28 gray scale (1 channel) images, i.e., $\mathcal{X} = \mathbb{R}^{28\times 28}$ or $\mathbb{R}^{784}$
- Output: digits 0 through 9 (i.e., $\mathcal{Y} = \{0, 1, \ldots, 9\}$)

CIFAR datasets



- Input: 32×32 RGB color (3 channels) images, i.e., $\mathcal{X} = \mathbb{R}^{32 \times 32 \times 3}$ or $\mathbb{R}^{3072}$
- Output: 10 classes (airplanes, cars, birds, cats, deer, dogs, frogs, horses, ships, and trucks) or 100 classes

ImageNet dataset



- Input: varies, often high-resolution (often $224 \times 224 \times 3$)
- Output: 1000 different categories

Mathematical set-up

- Modeling assumption: the data (input-output pairs) come from an underlying data distribution ρ over $\mathcal{X} \times \mathcal{Y}$
- Training data: $(x_1, y_1), \ldots, (x_n, y_n) \overset{\text{i.i.d.}}{\sim} \rho$
- Error metric: for any given classifier f, its risk, defined as the average (expected) classification error on a new data is

$$R(f) := \mathbb{P}_{(X,Y)\sim\rho}(f(X) \neq Y)$$

- Supervised learning: build a classifier f based on training data, that makes the average classification error as small as possible

Questions

- Does there exists a "best" classifier?

  — *this lecture*

- Can we construct this "best" classifier with the information of ρ?

  — *this lecture*

- What can we do when we only have a finite number of training data?

  — *first half of the semester*

Bayes optimal classifier: binary case

- Consider the binary case: $\mathcal{Y} = \{0, 1\}$
- Define the Bayes classifier: for any $x \in \mathcal{X}$,

$$f^\star(x) := \begin{cases} 1, & \text{if } \mathbb{P}(Y = 1 \mid X = x) \geq \mathbb{P}(Y = 0 \mid X = x), \\ 0, & \text{otherwise.} \end{cases}$$

Theorem 2.1 (Bayes optimal classifier: binary case)

The Bayes classifier $f^\star$ minimizes the misclassification error, i.e.,

$$f^\star \in \operatorname*{arg\,min}_{f:\mathcal{X}\to\mathcal{Y}} \mathbb{P}_{(X,Y)\sim\rho}(f(X) \neq Y).$$

Proof of Theorem 2.1

We need to show that, for any classifier $f : \mathcal{X} \to \mathcal{Y}$,

$$R(f) = \mathbb{P}(f(X) \neq Y) \geq \mathbb{P}(f^\star(X) \neq Y) = R(f^\star)$$

By tower property,

$$\begin{aligned}
\mathbb{P}(f(X) \neq Y) &= \mathbb{E}\left[\mathbb{1}_{f(X)\neq Y}\right] \\
&= \mathbb{E}_X\left[\mathbb{E}\left[\mathbb{1}_{f(X)\neq Y} \mid X\right]\right] && \text{(tower property)} \\
&= \mathbb{E}_X\left[\mathbb{P}\left(f(X) \neq Y \mid X\right)\right] \\
&\geq \mathbb{E}_X\left[\mathbb{P}\left(f^\star(X) \neq Y \mid X\right)\right] && \text{(why?)} \\
&= \mathbb{E}_X\left[\mathbb{E}\left[\mathbb{1}_{f^\star(X)\neq Y} \mid X\right]\right] \\
&= \mathbb{E}\left[\mathbb{1}_{f^\star(X)\neq Y}\right] && \text{(tower property)} \\
&= \mathbb{P}(f^\star(X) \neq Y).
\end{aligned}$$

It suffices to check

$$\mathbb{P}\left(f(X) \neq Y \mid X\right) \geq \mathbb{P}\left(f^\star(X) \neq Y \mid X\right).$$

Proof of Theorem 2.1 (cont.)

Observe that

$$\mathbb{P}(f^\star(X) \neq Y \mid X) = \begin{cases} \mathbb{P}(Y=0 \mid X) & \text{if } \mathbb{P}(Y=1 \mid X) \geq \mathbb{P}(Y=0 \mid X) \\ \mathbb{P}(Y=1 \mid X) & \text{if } \mathbb{P}(Y=1 \mid X) \geq \mathbb{P}(Y=0 \mid X) \end{cases}$$
$$= \min\left\{\mathbb{P}(Y=1 \mid X), \mathbb{P}(Y=0 \mid X)\right\}$$

and

$$\mathbb{P}(f(X) \neq Y \mid X) = \begin{cases} \mathbb{P}(Y=0 \mid X) & \text{if } f(X)=1 \\ \mathbb{P}(Y=1 \mid X) & \text{if } f(X)=0 \end{cases}$$
$$\geq \min\left\{\mathbb{P}(Y=1 \mid X), \mathbb{P}(Y=0 \mid X)\right\}.$$

Therefore

$$\mathbb{P}(f^\star(X) \neq Y \mid X) \geq \mathbb{P}(f(X) \neq Y \mid X).$$

A few remarks

Bayes optimal classifier

$$f^\star(x) := \begin{cases} 1, & \text{if } \mathbb{P}(Y = 1 \mid X = x) \geq \mathbb{P}(Y = 0 \mid X = x), \\ 0, & \text{otherwise.} \end{cases}$$

- Depends on the true underlying data distribution ρ
- The optimal classifier might not be unique
- When $\mathcal{X}$ is discrete, it is equivalent to

$$f^\star(x) := \begin{cases} 1, & \text{if } \mathbb{P}(X = x, Y = 1) \geq \mathbb{P}(X = x, Y = 0), \\ 0, & \text{otherwise.} \end{cases}$$

Bayes risk: binary case

- Bayes risk:

$$R^\star := \mathbb{P}_{(X,Y)\sim\rho}(f^\star(X) \neq Y)$$

- The Bayes risk serves as a lower bound for the classification error that any practical classifier can achieve:

$$R^\star = \min_{f:\mathcal{X}\to\mathcal{Y}} \mathbb{P}_{(X,Y)\sim\rho}(f(X) \neq Y).$$

- It represents the inherent uncertainty in the classification problem due to overlapping distributions of the classes.
- Excess risk: $R(f) - R^\star$

Bayes optimal classifier: multiclass setting

- Consider the multiclass case: $\mathcal{Y} = \{1, \ldots, K\}$
- Define the Bayes classifier: for any $x \in \mathcal{X}$,

$$f^\star(x) := \arg\max_{y \in \mathcal{Y}} \mathbb{P}(Y = y \mid X = x)$$

Theorem 2.2 (Bayes optimal classifier: multiclass case)

The Bayes classifier $f^\star$ minimizes the misclassification error, i.e.,

$$f^\star \in \arg\min_{f : \mathcal{X} \to \mathcal{Y}} \mathbb{P}_{(X,Y) \sim \rho}(f(X) \neq Y).$$

Bayes optimal classifier: multiclass setting

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The Bayes classifier $f^\star$ minimizes the misclassification error, i.e.,

$$f^\star \in \arg\min_{f : \mathcal{X} \to \mathcal{Y}} \mathbb{P}_{(X,Y)\sim\rho}(f(X) \neq Y).$$

Proof: similar to Theorem 2.1, it suffices to check for any classifier f

$$\mathbb{P}\left(f(X) \neq Y \mid X\right) \geq \mathbb{P}\left(f^\star(X) \neq Y \mid X\right).$$

More general loss function?

- Consider more general loss function $\ell : \mathcal{Y} \times \mathcal{Y} \to \mathbb{R}$
- Define the risk for a classifier $f : \mathcal{X} \to \mathcal{Y}$ as

$$R_\ell(f) := \mathbb{E}_{(X,Y)\sim\rho}[\ell(f(X), Y)]$$

- Example: with 0-1 loss $\ell(y, y') = \mathbb{1}\{y \neq y'\}$, we recover the average classification error

$$R(f) = \mathbb{P}_{(X,Y)\sim\rho}(f(X) \neq Y)$$

- Goal: find f that minimizes the risk $R_\ell(f)$ (the Bayes classifier might not be optimal...)

Question: Can you think of settings where other types of loss functions are more appropriate than the 0-1 loss?

Example: traffic signs

- $\mathcal{Y} = \{\text{stop sign}, 50 \text{ mph}, 40 \text{ mph}\}$.
- Predicting 50 mph when it is actually a stop sign is worse than predicting 40 mph when it is actually 50mph.
- 0-1 loss is not suitable here...

Example: traffic signs

- $\mathcal{Y} = \{\text{stop sign}, 50 \text{ mph}, 40 \text{ mph}\}$.
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- 0-1 loss is not suitable here...

We will discuss classification with general loss later if time permits

Supervised learning

- Go back to 0-1 loss
- In practice, we don't know ρ. It is in general impossible to compute the Bayes classifier $f^\star$
- Goal: build a classifier $f : \mathcal{X} \to \mathcal{Y}$ based on training data $(x_1, y_1), \ldots, (x_n, y_n) \overset{\text{i.i.d.}}{\sim} \rho$
- Hope: achieve small excess risk $R(f) - R^\star$
- High-level framework:
  - Make some modeling assumptions on ρ
  - Design a good classifier f under this setup
  - For example, a good classifier may satisfy

$$R(f) - R^\star \leq h(n)$$

where $h(n)$ is a function of the sample size n describing the rate of convergence, e.g., $h(n) = O(1/n)$.

Linear Methods for Classification

Linear classifiers

- Linear classifiers: decision boundaries are linear hyperplanes
  - Hyperplane $\mathcal{H}_{\boldsymbol{\beta},\beta_0} = \{\boldsymbol{x} \in \mathbb{R}^d : \langle \boldsymbol{\beta}, \boldsymbol{x} \rangle + \beta_0 = 0\}$
  - Half planes cut by $\mathcal{H}_{\boldsymbol{\beta},\beta_0}$:

$$\mathcal{H}^+_{\boldsymbol{\beta},\beta_0} = \{\boldsymbol{x} \in \mathbb{R}^d : \langle \boldsymbol{\beta}, \boldsymbol{x} \rangle + \beta_0 \geq 0\},$$
$$\mathcal{H}^-_{\boldsymbol{\beta},\beta_0} = \{\boldsymbol{x} \in \mathbb{R}^d : \langle \boldsymbol{\beta}, \boldsymbol{x} \rangle + \beta_0 < 0\}.$$

  - Example: in the binary case, the linear classifier has the form

$$f(\boldsymbol{x}) = \mathbb{1}\{\boldsymbol{x} \in \mathcal{H}^+_{\boldsymbol{\beta},\beta_0}\}$$

- Three approaches to learn a linear classifier from the data:
  - Linear discriminant analysis (LDA)
  - Logistic regression
  - Perceptrons and Support vector machines (SVMs)

Linear discriminant analysis (LDA)

- Model set-up: $\mathcal{X} = \mathbb{R}^d$, $\mathcal{Y} = \{1, \ldots, K\}$. For $k = 1, \ldots, K$,

$$\mathbb{P}(Y = k) = \omega_k, \qquad X \mid Y = k \sim \mathcal{N}(\boldsymbol{\mu}_k, \boldsymbol{\Sigma})$$

  where $\omega_k \geq 0$, $\sum_{k=1}^K \omega_k = 1$, $\boldsymbol{\mu}_k \in \mathbb{R}^d$, $\boldsymbol{\Sigma} \in \mathbb{S}_+^d$

- The Bayes classifier under this setup: for any $\boldsymbol{x}$, compute

$$\delta_k(\boldsymbol{x}) := \underbrace{\boldsymbol{x}^\top \boldsymbol{\Sigma}^{-1} \boldsymbol{\mu}_k - \frac{1}{2} \boldsymbol{\mu}_k^\top \boldsymbol{\Sigma}^{-1} \boldsymbol{\mu}_k + \log \omega_k}_{\propto \log \mathbb{P}(Y = k \mid X = \boldsymbol{x}) + \text{constant}} .$$

  Let $f^\star(\boldsymbol{x}) = \arg\max_{1 \leq k \leq K} \delta_k(\boldsymbol{x})$.

- Issue: model parameters are unknown...

Plug-in approach

- Plug-in approach: replace the unknown parameters with reliable estimates
- Suppose we have i.i.d. data $(\boldsymbol{x}_1, y_1), \ldots, (\boldsymbol{x}_n, y_n) \overset{\text{i.i.d.}}{\sim} \rho$
- For each $1 \le k \le K$, let $n_k = \sum_{i=1}^n \mathbb{1}\{y_i = k\}$ and

$$\widehat{\boldsymbol{\mu}}_k = \frac{1}{n_k} \sum_{i:y_i=k} \boldsymbol{x}_i, \qquad \widehat{\omega}_k = \frac{n_k}{n}$$

- Estimate the covariance matrix

$$\widehat{\boldsymbol{\Sigma}} = \frac{1}{N - K} \sum_{k=1}^{K} \sum_{i:y_i=k} \left(\boldsymbol{x}_i - \widehat{\boldsymbol{\mu}}_k\right)\left(\boldsymbol{x}_i - \widehat{\boldsymbol{\mu}}_k\right)^\top$$

- Replace $\boldsymbol{\mu}_k$, ω_k, $\boldsymbol{\Sigma}$ with $\widehat{\boldsymbol{\mu}}_k$, $\widehat{\omega}_k$, $\widehat{\boldsymbol{\Sigma}}$

$$\widehat{\delta}_k(\boldsymbol{x}) := \underbrace{\boldsymbol{x}^\top \widehat{\boldsymbol{\Sigma}}^{-1} \widehat{\boldsymbol{\mu}}_k - \frac{1}{2} \widehat{\boldsymbol{\mu}}_k^\top \widehat{\boldsymbol{\Sigma}}^{-1} \widehat{\boldsymbol{\mu}}_k + \log \widehat{\omega}_k}_{\text{linear in } \boldsymbol{x}}.$$

Generalization

- Consider a more general set-up: for $k = 1, \ldots, K$, assume

$$\mathbb{P}(Y = k) = \omega_k, \qquad X \mid Y = k \sim \mathcal{N}(\boldsymbol{\mu}_k, \boldsymbol{\Sigma}_k)$$

  where $\omega_k \geq 0$, $\sum_{k=1}^{K} \omega_k = 1$, $\mu_k \in \mathbb{R}^d$, $\boldsymbol{\Sigma}_k \in \mathbb{S}_+^d$

- This setup will lead to the so-called quadratic discriminant analysis (QDA)
- Homework: derive QDA
  - What is the Bayes classifier under this setup?
  - How to derive a practical (data-driven) classifier?
  - Is this still a linear classifier?

Logistic regression

- Model set-up: $\mathcal{X} = \mathbb{R}^d$, $\mathcal{Y} = \{0, 1, \ldots, K\}$. Let

$$\mathbb{P}(Y = k \mid \boldsymbol{x}) = \frac{\exp(\boldsymbol{\beta}_k^\top \boldsymbol{x} + \beta_{0,k})}{1 + \sum_{k'=1}^{K} \exp(\boldsymbol{\beta}_{k'}^\top \boldsymbol{x} + \beta_{0,k'})}, \quad (1 \leq k \leq K),$$

$$\mathbb{P}(Y = 0 \mid \boldsymbol{x}) = \frac{1}{1 + \sum_{k'=1}^{K} \exp(\boldsymbol{\beta}_{k'}^\top \boldsymbol{x} + \beta_{0,k})},$$

  where the parameters $\boldsymbol{\beta}_k \in \mathbb{R}^d$, $\beta_{0,k} \in \mathbb{R}$ for $k = 1, \ldots, K$

Logistic regression

- Model set-up: $\mathcal{X} = \mathbb{R}^d \times \{1\}$, $\mathcal{Y} = \{0, 1, \ldots, K\}$. Let

$$\mathbb{P}(Y = k \mid \boldsymbol{x}) = \frac{\exp(\boldsymbol{\beta}_k^\top \boldsymbol{x})}{1 + \sum_{k'=1}^K \exp(\boldsymbol{\beta}_{k'}^\top \boldsymbol{x})}, \qquad (k = 1, \ldots, K),$$
$$\mathbb{P}(Y = 0 \mid \boldsymbol{x}) = \frac{1}{1 + \sum_{k'=1}^K \exp(\boldsymbol{\beta}_{k'}^\top \boldsymbol{x})},$$

  where the parameters $\boldsymbol{\beta}_k \in \mathbb{R}^{d+1}$ for $k = 1, \ldots, K$

- Bayes classifier:

$$f(\boldsymbol{x}) = \begin{cases} \text{argmax}_{1 \le k \le K} \boldsymbol{\beta}_k^\top \boldsymbol{x}, & \text{if } \max_{1 \le k \le K} \boldsymbol{\beta}_k^\top \boldsymbol{x} > 0, \\ 0, & \text{otherwise.} \end{cases}$$

- Estimate $\boldsymbol{\beta}_k$'s: maximum likelihood estimation (MLE)

Maximum likelihood estimation

- Suppose we have i.i.d. data $(\boldsymbol{x}_1, y_1), \ldots, (\boldsymbol{x}_n, y_n)$
- The negative log-likelihood function

$$\ell(\boldsymbol{\beta}) = -\frac{1}{n} \sum_{k=1}^{K} \sum_{i:y_i=k} \boldsymbol{x}_i^\top \boldsymbol{\beta}_k + \frac{1}{n} \sum_{i=1}^{n} \log \left[1 + \sum_{k'=1}^{K} \exp(\boldsymbol{x}_i^\top \boldsymbol{\beta}_{k'})\right]$$

- Maximum likelihood estimation (MLE)

$$\widehat{\boldsymbol{\beta}} := \arg\min_{\boldsymbol{\beta}} \ell(\boldsymbol{\beta})$$

- Convex optimization: solve by e.g., gradient descent

$$\boldsymbol{\beta}^{t+1} = \boldsymbol{\beta}^t - \eta \nabla \ell(\boldsymbol{\beta}^t) \qquad (t = 0, 1, \ldots)$$

A brief introduction to gradient descent

Gradient descent (GD) for solving $\min_{\boldsymbol{\beta}\in\mathbb{R}^d} L(\boldsymbol{\beta})$:

$$\boldsymbol{\beta}^{t+1} = \boldsymbol{\beta}^t - \eta\nabla L(\boldsymbol{\beta}^t) \qquad (t = 0, 1, \ldots)$$

When η is properly small, GD satisfy the following properties:

- For smooth function L, GD is a descent algorithm: $L(\boldsymbol{\beta}^{t+1}) \leq L(\boldsymbol{\beta}^t)$
- For convex + smooth function L, GD satisfies

  $$L(\boldsymbol{\beta}^t) - L(\boldsymbol{\beta}^\star) \leq O\left(\frac{\|\boldsymbol{\beta}^0 - \boldsymbol{\beta}^\star\|_2^2}{t}\right) \qquad (t = 0, 1, \ldots)$$

  for any minimizer $\boldsymbol{\beta}^\star$
- For strongly convex + smooth function L, GD satisfies

  $$\|\boldsymbol{\beta}^{t+1} - \boldsymbol{\beta}^\star\|_2 \leq (1-\kappa)^t\|\boldsymbol{\beta}^0 - \boldsymbol{\beta}^\star\|_2 \qquad (t = 0, 1, \ldots)$$

  for some $\kappa \in (0, 1)$, where $\boldsymbol{\beta}^\star$ is the unique minimizer

Stochastic gradient descent

Consider the following empirical risk minimization problem

$$\min_{\boldsymbol{\beta} \in \mathbb{R}^d} L(\boldsymbol{\beta}) := \frac{1}{n} \sum_{i=1}^{n} g(\boldsymbol{\beta}; \boldsymbol{x}_i),$$

where $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n$ are training data points.

- **Stochastic gradient descent:** for $t = 0, 1, \ldots,$

$$\boldsymbol{\beta}^{t+1} = \boldsymbol{\beta}^t - \eta \nabla g(\boldsymbol{\beta}^t; \boldsymbol{x}_{i_t}) \quad \text{where} \quad \boldsymbol{x}_{i_t} \overset{\text{ind.}}{\sim} \text{Unif}\{\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n\}$$

- **Gradient descent:** for $t = 0, 1, \ldots,$

$$\boldsymbol{\beta}^{t+1} = \boldsymbol{\beta}^t - \eta \nabla L(\boldsymbol{\beta}^t) = \boldsymbol{\beta}^t - \eta \frac{1}{n} \sum_{i=1}^{n} \nabla g(\boldsymbol{\beta}; \boldsymbol{x}_i)$$

- **Advantage of SGD:** much faster updates, especially for large datasets, but still enjoys nice properties (sometimes even better than GD!)

Gradient descent methods

Example: GD / SGD for logistic regresion

Take-away: (stochastic) gradient descent is the default method for solving unconstrained optimization problem

— simple and effective!

Recommended reading materials: Lecture 1 and 10 of the course

Large-Scale Optimization for Data Science

by Prof. Yuxin Chen (UPenn); Lecture on GD and SGD

Perceptrons and SVMs

Linearly separable data

- Consider binary classification: $\mathcal{X} = \mathbb{R}^d$ and $\mathcal{Y} = \{1, -1\}$
- Training data: $(\boldsymbol{x}_1, y_1), \ldots, (\boldsymbol{x}_n, y_n)$
- **Linearly separable data:** $\exists$ a separating hyperplane $\mathcal{H}_{\boldsymbol{\beta},\beta_0}$ s.t.

$$y_i \cdot (\boldsymbol{x}_i^\top \boldsymbol{\beta} + \beta_0) > 0 \qquad (i = 1, \ldots, n)$$

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- by merging β_0 into $\boldsymbol{\beta}$ and adding 1 to $\boldsymbol{x}_i$'s, this assumption becomes: $\exists\, \boldsymbol{\beta}_{\text{sep}} \in \mathbb{R}^{d+1}$

$$y_i \cdot \boldsymbol{x}_i^\top \boldsymbol{\beta}_{\text{sep}} > 0 \qquad (i = 1, \ldots, n)$$

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$$y_i \cdot \boldsymbol{x}_i^\top \boldsymbol{\beta}_{\text{sep}} > 0 \qquad (i = 1, \ldots, n)$$

- **Goal:** search a separating hyperplane indexed by $\widehat{\boldsymbol{\beta}}$

$$y_i \cdot \boldsymbol{x}_i^\top \widehat{\boldsymbol{\beta}} > 0 \qquad (i = 1, \ldots, n)$$

(note that $\boldsymbol{\beta}_{\text{sep}}$ is not known a priori)

Perceptron Learning Algorithm

- For every $\boldsymbol{\beta} \in \mathbb{R}^{d+1}$, define the set $\mathcal{M}_{\boldsymbol{\beta}} \coloneqq \underbrace{\{i : y_i \cdot \boldsymbol{x}_i^\top \boldsymbol{\beta} \leq 0\}}_{\text{misclassified points}}$
- Target: minimize the perceptron loss

$$\sigma(\boldsymbol{\beta}) \coloneqq -\sum_{i \in \mathcal{M}_{\boldsymbol{\beta}}} y_i \cdot \boldsymbol{x}_i^\top \boldsymbol{\beta} \propto \sum_{i \in \mathcal{M}_{\boldsymbol{\beta}}} \text{dist}(\boldsymbol{x}_i, \mathcal{H}_{\boldsymbol{\beta}})$$

  where $\mathcal{H}_{\boldsymbol{\beta}} = \{\boldsymbol{x} : \boldsymbol{x}^\top \boldsymbol{\beta} = 0\}$
- Algorithm: initialize with $\boldsymbol{\beta}^0 \in \mathbb{R}^{d+1}$, for $t = 0, 1, \ldots,$ update

$$\boldsymbol{\beta}^{t+1} = \boldsymbol{\beta}^t + \eta y_i \boldsymbol{x}_i, \quad \text{for a random } i \in \mathcal{M}_{\boldsymbol{\beta}^t}$$

  where $\eta > 0$ is the step size; in fact, we can take $\eta = 1$ here...

Perceptron Learning Algorithm

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- Algorithm: initialize with $\boldsymbol{\beta}^0 \in \mathbb{R}^{d+1}$, for $t = 0, 1, \ldots,$ update

$$\boldsymbol{\beta}^{t+1} = \boldsymbol{\beta}^t + y_i \boldsymbol{x}_i, \quad \text{for a random } i \in \mathcal{M}_{\boldsymbol{\beta}^t}$$

- Interpretation: SGD with step size 1 (kind of...)

Convergence theory

Theorem 2.3

When the data is linearly separable, the perceptron learning algorithm converges to a separating hyperplane in a finite number of steps.

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- only works linearly separable data. If the classes cannot be separated by a hyperplane, the algorithm will not converge

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- solutions not unique: might converge to an unstable hyperplane
  — resort to "optimal separating hyperplane"
- only works linearly separable data. If the classes cannot be separated by a hyperplane, the algorithm will not converge
- the "finite" number of steps can be very large

Optimal separating hyperplane

From now on, we "unmerge" β_0 from $\boldsymbol{\beta}$, as they play different roles. Consider the optimization problem

$$\max_{\|\boldsymbol{\beta}\|_2=1,\beta_0,M} \quad M \quad \text{s.t.} \quad y_i(\boldsymbol{x}_i^\top \boldsymbol{\beta} + \beta_0) \geq M \quad (i = 1, \ldots, n)$$

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Implications:

- the distance between $\boldsymbol{x}$ and the hyperplane $\mathcal{H}_{\boldsymbol{\beta},\beta_0}$ is

$$\text{dist}(\boldsymbol{x}, \mathcal{H}_{\boldsymbol{\beta},\beta_0}) = \frac{|\boldsymbol{\beta}^\top \boldsymbol{x} + \beta_0|}{\|\boldsymbol{\beta}\|_2} \overset{\text{if } \|\beta\|_2=1}{=\!=} |\boldsymbol{\beta}^\top \boldsymbol{x} + \beta_0|$$

Optimal separating hyperplane

From now on, we "unmerge" β_0 from $\boldsymbol{\beta}$, as they play different roles. Consider the optimization problem

$$\max_{\|\boldsymbol{\beta}\|_2=1,\beta_0,M} \quad M \quad \text{s.t.} \quad y_i(\boldsymbol{x}_i^\top \boldsymbol{\beta} + \beta_0) \geq M \quad (i = 1, \ldots, n)$$

Implications:

- the distance between $\boldsymbol{x}$ and the hyperplane $\mathcal{H}_{\boldsymbol{\beta},\beta_0}$ is $|\boldsymbol{\beta}^\top \boldsymbol{x} + \beta_0|$
- offers a unique solution that maximizes the *margin* M
- **Margin:** the distance between $\mathcal{H}_{\boldsymbol{\beta},\beta_0}$ and the $\underbrace{\text{closest data points}}_{\text{support vectors}}$ from each class

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- offers a unique solution that maximizes the *margin* M
- **Margin:** the distance between $\mathcal{H}_{\boldsymbol{\beta},\beta_0}$ and the $\underbrace{\text{closest data points}}_{\text{support vectors}}$ from each class
- **Intuition:** a large margin on the training data will lead to good separation on the test data.

Reformulation as convex optimization

- **Original problem:**

$$\max_{\|\boldsymbol{\beta}\|_2=1,\beta_0,M} \quad M \quad \text{s.t.} \quad y_i(\boldsymbol{x}_i^\top \boldsymbol{\beta} + \beta_0) \geq M \quad (i = 1, \ldots, n)$$

- **Issue:** this is not a convex optimization problem...

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- **Issue:** this is not a convex optimization problem...

- **Reformulation:**

$$\min_{\boldsymbol{\beta},\beta_0} \quad \|\boldsymbol{\beta}\|_2^2 \quad \text{s.t.} \quad y_i(\boldsymbol{x}_i^\top \boldsymbol{\beta} + \beta_0) \geq 1 \quad (i = 1, \ldots, n)$$

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Reformulation as convex optimization

- **Original problem:**

$$\max_{\|\boldsymbol{\beta}\|_2=1,\beta_0,M} \quad M \quad \text{s.t.} \quad y_i(\boldsymbol{x}_i^\top \boldsymbol{\beta} + \beta_0) \geq M \quad (i = 1, \ldots, n)$$

- **Issue:** this is not a convex optimization problem...
- **Reformulation:**

$$\min_{\boldsymbol{\beta},\beta_0} \quad \|\boldsymbol{\beta}\|_2^2 \quad \text{s.t.} \quad y_i(\boldsymbol{x}_i^\top \boldsymbol{\beta} + \beta_0) \geq 1 \quad (i = 1, \ldots, n)$$

  this is a convex optimization problem

- This is known as the support vector machine (SVM)

SVMs for separable data

$$\min_{\boldsymbol{\beta},\beta_0} \quad \frac{1}{2}\|\boldsymbol{\beta}\|_2^2 \quad \text{s.t.} \quad y_i(\boldsymbol{x}_i^\top \boldsymbol{\beta} + \beta_0) \geq 1 \quad (i = 1, \ldots, n)$$

- SVM is a powerful method for binary classification
- finds a linear classifier with decision boundary $\{\boldsymbol{x} : \boldsymbol{x}^\top \widehat{\boldsymbol{\beta}} + \widehat{\beta}_0 = 0\}$ to separate two classes with the maximum margin
- This is only feasible for *linearly separated data*

  — *can be generalized to accommodate non-separable data*
- What can we say about SVM?

  — *resort to duality theory!*

Convex optimization and duality theory

Primal problem and Lagrangian function

- Consider a convex optimization problem:

$$\min_{\boldsymbol{x} \in \mathbb{R}^d} f(\boldsymbol{x}) \quad \text{s.t.} \quad g_i(\boldsymbol{x}) \leq 0 \quad (i = 1, \ldots, m).$$

  where $f(\boldsymbol{x})$ and $g_i(\boldsymbol{x})$ are convex functions

- This is called the **primal problem**
- To handle the constraints, we introduce **Lagrange multipliers** λ_i
- The Lagrangian function is:

$$L(\boldsymbol{x}, \boldsymbol{\lambda}) = f(\boldsymbol{x}) + \sum_{i=1}^{m} \lambda_i g_i(\boldsymbol{x})$$

- What is the benefit of introducing the Lagrangian function?

The Dual Problem

Key observation:

$$\underbrace{\min_{\boldsymbol{x}:g(\boldsymbol{x})\leq 0} f(\boldsymbol{x})}_{\text{primal problem}} \overset{\text{(i)}}{=} \min_{\boldsymbol{x}} \max_{\boldsymbol{\lambda}\geq 0} L(\boldsymbol{x},\boldsymbol{\lambda}) \overset{\text{(ii)}}{\geq} \max_{\boldsymbol{\lambda}\geq 0} \underbrace{\min_{\boldsymbol{x}} L(\boldsymbol{x},\boldsymbol{\lambda})}_{=:d(\boldsymbol{\lambda})} = \underbrace{\max_{\boldsymbol{\lambda}\geq 0} d(\boldsymbol{\lambda})}_{\text{dual problem}}$$

- relation (i) and (ii) always holds (why?)
- relation (ii) is often an equality (strong duality theory)
- The dual function $d(\boldsymbol{\lambda}) = \min_{\boldsymbol{x}} L(\boldsymbol{x},\boldsymbol{\lambda})$
- The **dual problem** is to maximize the dual function $d(\boldsymbol{\lambda})$:

$$\max_{\boldsymbol{\lambda}\geq 0} d(\boldsymbol{\lambda})$$

Strong and Weak Duality

Weak Duality: For any $\boldsymbol{x}$ feasible in the primal and any $\boldsymbol{\lambda} \geq 0$, we have:

$$d(\boldsymbol{\lambda}) \leq f(\boldsymbol{x})$$

Strong Duality: If the problem satisfies certain conditions (e.g., Slater's condition), then:

$$\min_{\boldsymbol{x}: g(\boldsymbol{x}) \leq 0} f(\boldsymbol{x}) = \max_{\boldsymbol{\lambda} \geq 0} d(\boldsymbol{\lambda})$$

- Slater's condition: the feasible region has an interior point, i.e.,

$$\exists\, \boldsymbol{x}_0 \in \mathbb{R}^d \quad \text{s.t.} \quad g_i(\boldsymbol{x}_0) < 0 \quad (i = 1, \ldots, m).$$

- In convex optimization, strong duality often holds, meaning the primal and dual problems have the same optimal value.

KKT Conditions

The **Karush-Kuhn-Tucker (KKT)** conditions: if strong duality holds, and $(\boldsymbol{x}, \boldsymbol{\lambda})$ is the optimal solution pair for the primal/dual problem

$$\underbrace{\min_{\boldsymbol{x}: g(\boldsymbol{x}) \leq 0} f(\boldsymbol{x})}_{\text{primal problem}} = \underbrace{\max_{\boldsymbol{\lambda} \geq 0} d(\boldsymbol{\lambda})}_{\text{dual problem}},$$

then

- **Primal feasibility:** $g_i(\boldsymbol{x}) \leq 0$
- **Dual feasibility:** $\lambda_i \geq 0$
- **Complementary slackness:** $\lambda_i g_i(\boldsymbol{x}) = 0$
- **Stationarity:** $\nabla f(\boldsymbol{x}) + \sum_{i=1}^{m} \lambda_i \nabla g_i(\boldsymbol{x}) = 0$

— This is a necessary condition!

Back to SVMs

$$\min_{\boldsymbol{\beta},\beta_0} \quad \frac{1}{2}\|\boldsymbol{\beta}\|_2^2 \quad \text{s.t.} \quad y_i(\boldsymbol{x}_i^\top \boldsymbol{\beta} + \beta_0) \geq 1 \quad (i = 1, \ldots, n)$$

- The **dual problem** for SVM is (why?):

$$\max_{\alpha} \sum_{i=1}^{n} \alpha_i - \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} \alpha_i \alpha_j y_i y_j \boldsymbol{x}_i^\top \boldsymbol{x}_j \quad \text{s.t.} \quad \sum_{i=1}^{n} \alpha_i y_i = 0,\ \alpha_i \geq 0$$

- It is straightforward to check that Slater's condition holds

  — *primal and dual problems are equivalent!*

- The dual problem is a quadratic programming problem, which is easier to compute with standard software (e.g. CVX)

Checking KKT conditions

$$\text{(P)} \quad \min_{\boldsymbol{\beta},\beta_0} \quad \frac{1}{2}\|\boldsymbol{\beta}\|_2^2 \quad \text{s.t.} \quad y_i(\boldsymbol{x}_i^\top \boldsymbol{\beta} + \beta_0) \geq 1 \quad (i = 1, \ldots, n)$$

$$\text{(D)} \quad \max_{\alpha} \sum_{i=1}^n \alpha_i - \frac{1}{2}\sum_{i=1}^n \sum_{j=1}^n \alpha_i \alpha_j y_i y_j \boldsymbol{x}_i^\top \boldsymbol{x}_j \quad \text{s.t.} \quad \sum_{i=1}^n \alpha_i y_i = 0,\ \alpha_i \geq 0$$

The Karush-Kuhn-Tucker (KKT) conditions for optimality:

- **Primal feasibility:** $y_i(\boldsymbol{\beta}^\top \boldsymbol{x}_i + \beta_0) \geq 1$
- **Dual feasibility:** $\alpha_i \geq 0$
- **Complementary slackness:** $\alpha_i[y_i(\boldsymbol{\beta}^\top \boldsymbol{x}_i + \beta_0) - 1] = 0$
- **Stationarity:** $\boldsymbol{\beta} = \sum_{i=1}^n \alpha_i y_i \boldsymbol{x}_i$

Implications

For any optimal solution pair $(\boldsymbol{\beta}^\star, \beta_0^\star, \alpha^\star)$:

- **Support vectors:** data points $\boldsymbol{x}_i$ with $\alpha_i > 0$

$$\begin{aligned} y_i(\boldsymbol{\beta}^{\star\top}\boldsymbol{x}_i + \beta_0^\star) > 1 \quad &\Longrightarrow \quad \alpha_i = 0 \\ \alpha_i > 0 \quad &\Longrightarrow \quad y_i(\boldsymbol{\beta}^{\star\top}\boldsymbol{x}_i + \beta_0^\star) = 1 \end{aligned}$$

- **Recovering the primal solution:** after solving the dual problem (i.e., finding $\alpha_i^\star$), we can recover the primal solution $(\boldsymbol{\beta}^\star, \beta_0^\star)$ by

$$\boldsymbol{\beta}^\star = \sum_{i=1}^{n} \alpha_i^\star y_i \boldsymbol{x}_i$$

and $\beta_0^\star = y_i - \boldsymbol{\beta}^\top \boldsymbol{x}_i$ for any support vector $\boldsymbol{x}_i$

— *$\boldsymbol{\beta}^\star$ is a linear combination of the support vectors*

Accommodating non-separable data

SVM for linearly separable data:

$$\min_{\boldsymbol{\beta},\beta_0} \quad \frac{1}{2}\|\boldsymbol{\beta}\|_2^2 \quad \text{s.t.} \quad y_i(\boldsymbol{x}_i^\top \boldsymbol{\beta} + \beta_0) \geq 1 \quad (i = 1, \ldots, n)$$

- For non-separable data, we introduce slack variables $\xi_i \geq 0$ to allow violations of the margin:

$$\begin{aligned} \min_{\boldsymbol{\beta},\beta_0,\xi} \ & \frac{1}{2}\|\boldsymbol{\beta}\|^2 + C\sum_{i=1}^{n} \xi_i \\ \text{s.t.} \quad & y_i(\boldsymbol{\beta}^\top \boldsymbol{x}_i + \beta_0) \geq 1 - \xi_i, \quad \xi_i \geq 0 \quad (i = 1, \ldots, n) \end{aligned}$$

- $C > 0$ is the "cost" parameter
- the separable case corresponds to $C = \infty$

Dual problem: non-separable data

- **Primal problem:**

$$\min_{\boldsymbol{\beta},\beta_0,\xi} \ \frac{1}{2}\|\boldsymbol{\beta}\|^2 + C\sum_{i=1}^{n}\xi_i$$

$$\text{s.t.} \quad y_i(\boldsymbol{\beta}^\top \boldsymbol{x}_i + \beta_0) \geq 1 - \xi_i, \quad \xi_i \geq 0 \quad (i = 1, \ldots, n)$$

- **Dual problem:**

$$\max_{\alpha} \sum_{i=1}^{n}\alpha_i - \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}\alpha_i\alpha_j y_i y_j \boldsymbol{x}_i^\top \boldsymbol{x}_j$$

$$\text{s.t.} \quad \sum_{i=1}^{n}\alpha_i y_i = 0, \quad 0 \leq \alpha_i \leq C \quad (i = 1, \ldots, n)$$

- Homework: derive the dual problem from the primal problem

Dual problem: non-separable data

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- **Dual problem:**

$$\max_{\alpha} \sum_{i=1}^{n}\alpha_i - \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}\alpha_i\alpha_j y_i y_j \boldsymbol{x}_i^\top \boldsymbol{x}_j$$

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Kernel density classifier and naive Bayes classifier

Recap: Bayes optimal classifier

Bayes optimal classifier: for any $x \in \mathcal{X}$, output

$$f^\star(x) := \arg\max_{y \in \mathcal{Y}} \mathbb{P}(Y = y \mid X = x)$$

- Issue: depends on unknown data distribution ρ

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- Bayes formula:

$$\mathbb{P}(Y = y \mid X = x) = \frac{\mathbb{P}(X = x \mid Y = y)\,\mathbb{P}(Y = y)}{\sum_{y' \in \mathcal{Y}} \mathbb{P}(X = x \mid Y = y')\,\mathbb{P}(Y = y')}$$

— *Is it possible to estimate these quantities?*

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$$\mathbb{P}(Y = y \mid X = x) = \frac{\widehat{\mathbb{P}}(X = x \mid Y = y)\, \widehat{\mathbb{P}}(Y = y)}{\sum_{y' \in \mathcal{Y}} \widehat{\mathbb{P}}(X = x \mid Y = y')\, \widehat{\mathbb{P}}(Y = y')}$$

— *Is it possible to estimate these quantities?*

- Plug-in method:
  - marginal probabilities $\mathbb{P}(Y = y)$ are easy to estimate (use frequency)
  - key difficulty: estimate conditional densities $\mathbb{P}(X = x \mid Y = y)$

Detour: density estimation

Setup: density estimation

- Target: an unknown density function f
- What we have: i.i.d. data $X_1, \ldots, X_n \sim f$
- Goal: construct a good density estimation $\widehat{f}(\cdot)$ that satisfy

$$\widehat{f}(x) \geq 0 \quad \text{and} \quad \int_0^1 \widehat{f}(x) \mathrm{d}x = 1$$

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- Criteria: mean integrated squared error (MISE)

$$\mathsf{MISE}(\widehat{f}) = \mathbb{E}\Big[\int \big(\widehat{f}(x) - f(x)\big)^2 dx\Big]$$

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- Criteria: mean integrated squared error (MISE)

$$\mathrm{MISE}(\widehat{f}) = \mathbb{E}\left[\int \left(\widehat{f}(x) - f(x)\right)^2 dx\right]$$

- Density estimation: find $\widehat{f}$ with as small MISE as possible
  - Histogram method
  - Kernel density estimation

Bias-variance tradeoff

Mean integrated squared error (MISE):

$$\mathrm{MISE}(\widehat{f}) = \mathbb{E}\left[\int \left(\widehat{f}(x) - f(x)\right)^2 dx\right]$$

- **Bias:** Measures how far the estimated density is from the true density on average.

$$b(x) := \mathbb{E}[\widehat{f}(x)] - f(x)$$

- **Variance:** Measures how much $\hat{f}(x)$ fluctuates around its mean:

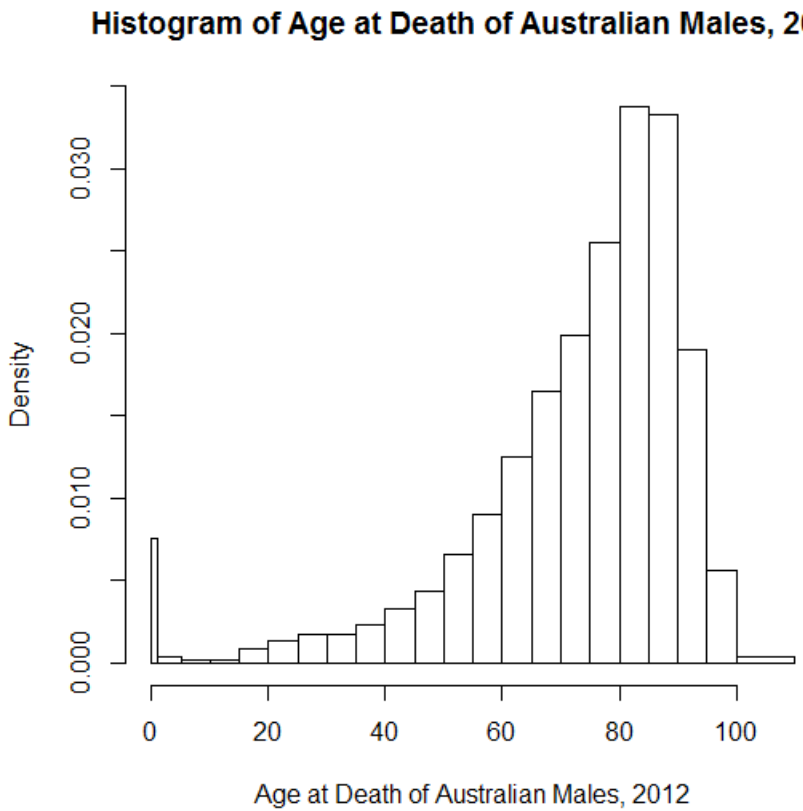
$$v(x) := \mathsf{var}(\widehat{f}(x)) = \mathbb{E}[(\widehat{f}(x) - \mathbb{E}[\widehat{f}(x)])^2]$$

Theorem 2.4

$$\mathrm{MISE}(\widehat{f}) = \int b^2(x)\mathrm{d}x + \int v(x)\mathrm{d}x$$

A simple approach: histogram

Histogram method: estimate the density by partitioning the interval and counting the frequency of data points in each partition



— *credit to R.J. Oosterbaan*

Histograms

- Consider 1D setting, and assume that $f(\cdot)$ is supported on $[0, 1]$

  — *we can always rescale the data to* $[0, 1]$

- The data is divided into m bins of equal width $h = 1/m$ (bandwidth)

$$B_1 = \left[0, \frac{1}{m}\right), \quad B_2 = \left[\frac{1}{m}, \frac{2}{m}\right), \quad \dots \quad , B_m = \left[\frac{m-1}{m}, 1\right]$$

- Each bin is assigned a probability proportional to the number of observations falling into that bin:

$$\widehat{f}(x) := \begin{cases} \widehat{p}_1/h, & x \in B_1, \\ \vdots & \vdots \\ \widehat{p}_m/h, & x \in B_m, \end{cases} \qquad \text{where} \quad \widehat{p}_j = \frac{1}{n}\sum_{i=1}^{n} \mathbb{1}\{X_i \in B_j\}.$$

Optimal bandwidth

Theorem 2.5 (informal)

Under some regularity conditions, we have

$$\mathrm{MISE}(\widehat{f}) \approx \frac{h^2}{12}\int f'(u)^2 \mathrm{d}u + \frac{1}{nh}$$

- *The optimal bandwidth choice is*

$$h^\star = \frac{1}{n^{1/3}}\left(\frac{6}{\int f'(u)^2 \mathrm{d}u}\right)^{1/3}$$

- *With this choice of $h^\star$, we have*

$$\mathrm{MISE}(\widehat{f}) \approx \frac{C}{n^{2/3}} \quad \textit{where} \quad C = \left(\frac{3}{4}\right)^{2/3}\left(\int f'(u)^2 \mathrm{d}u\right)^{1/3}.$$

Cross-validation

- Issue: the optimal bandwidth $h^\star$ depends on the unknown density f
- Idea: estimate the risk under each bandwidth selection h

$$L(h) := \int \left(\widehat{f}(x) - f(x)\right)^2 \mathrm{d}x = \underbrace{\int \widehat{f}^2(x)\mathrm{d}x - 2\int \widehat{f}(x)f(x)\mathrm{d}x}_{=:J(h)} + \int f^2(x)\mathrm{d}x$$

- Cross-validation estimate of the risk:

$$\widehat{J}(h) := \int \widehat{f}^2(x)\mathrm{d}x - \frac{2}{n}\sum_{i=1}^{n} \widehat{f}_{(-i)}(X_i)$$

- It can be shown that $\widehat{J}(h) \approx \mathbb{E}[J(h)]$
- **Cross validation:** select h that minimizes $\widehat{J}(h)$

Cross-validation

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- It can be shown that $\widehat{J}(h) \approx \mathbb{E}[J(h)]$
- **Cross validation:** select h that minimizes $\widehat{J}(h)$
- HW: prove the formula below that allows efficient computation of $\widehat{J}(h)$:

$$\widehat{J}(h) = \frac{2}{(n-1)h} - \frac{n+1}{n-1}\sum_{j=1}^{m} \widehat{p}_j^2$$

Limitation of the histogram method

- Histograms are discontinuous (not a continuous density)
- The convergence rate $O(n^{-2/3})$ is not ideal
- Complicated in higher dimension (number of bins will be exponential in dimension)
- A better solution: kernel density estimation

Kernel Density Estimation (KDE)

Kernel density estimator (KDE):

$$\widehat{f}(x) = \frac{1}{n}\sum_{i=1}^{n}\frac{1}{h}K\left(\frac{x - x_i}{h}\right)$$

where $K(\cdot)$ is a kernel function and $h > 0$ is the bandwidth

- **Kernel function:** any function $K(x) \geq 0$ that satisfies

$$\int K(x)\mathrm{d}x = 1, \quad \int xK(x)\mathrm{d}x = 0, \quad \int x^2K(x)\mathrm{d}x > 0$$

- Common kernel function:
  - Gaussian Kernel: $K(x) = \frac{1}{\sqrt{2\pi}}e^{-x^2/2}$
  - Epanechnikov kernel: $K(x) = \frac{3}{4\sqrt{5}}(1 - \frac{x^2}{5})\,\mathbb{1}\{|x| < \sqrt{5}\}$

Optimal bandwidth

Theorem 2.6

Under some regularity conditions, we have

$$R(f, \hat{f}_n) \approx \frac{h^4}{4}\Big(\int x^2 K(x)\mathrm{d}x\Big)^2 \int \left(f''(x)\right)^2 \mathrm{d}x + \frac{1}{nh}\int K^2(x)\mathrm{d}x.$$

The optimal bandwidth is

$$h^* = \frac{1}{n^{1/5}}\Big(\int x^2 K(x)dx\Big)^{-2/5}\Big(\int K^2(x)dx\Big)^{1/5}\Big(\int \left(f''(x)\right)^2 dx\Big)^{-1/5}$$

With this choice of bandwidth,

$$R(f, \hat{f}_n) \asymp \frac{1}{n^{4/5}}.$$

Cross-validation

Cross-validation: estimate the risk under each bandwidth selection h

$$L(h) := \int \big(\widehat{f}(x) - f(x)\big)^2 \mathrm{d}x = \underbrace{\int \widehat{f}^2(x)\mathrm{d}x - 2\int \widehat{f}(x)f(x)\mathrm{d}x}_{=:J(h)} + \int f^2(x)\mathrm{d}x$$

- Estimating $J(h)$:

$$\widehat{J}(h) := \int \widehat{f}^2(x)\mathrm{d}x - \frac{2}{n}\sum_{i=1}^{n} \widehat{f}_{(-i)}(X_i)$$

  It can be shown that $\mathbb{E}[\widehat{J}(h)] = \mathbb{E}[J(h)]$

- Cross validation: select h that minimizes $\widehat{J}(h)$
- An efficient formula for approximatly computing $\widehat{J}(h)$:

$$\widehat{J}(h) \approx \frac{1}{n^2}\sum_{i=1}^{n}\sum_{j=1}^{n} \frac{1}{h}K^*\big(\frac{X_i - X_j}{h}\big) + \frac{2}{nh}K(0)$$

  where $K^*(x) = \int K(x-y)K(y)\mathrm{d}y - 2K(x)$

Theoretical guarantees for cross validation

Theorem 2.7 (Stone's Theorem)

Suppose that f is bounded. Let $\widehat{f}_h$ be the KDE with bandwidth h, and let $\widehat{h}$ be the bandwidth chosen by cross-validation. Then

$$\frac{\mathsf{MISE}(\widehat{f}_{\widehat{h}})}{\inf_h \mathsf{MISE}(\widehat{f}_h)} \xrightarrow{\mathrm{P}} 1$$

as $n \to \infty$.

- Stone's theorem provides theoretical justification for using cross-validation to select bandwidth for KDE.

Implications

Faster convergence rate: $\underbrace{O(n^{-4/5})}_{\text{KDE}}$ vs. $\underbrace{O(n^{-2/3})}_{\text{histogram}}$

Extension to higher dimension: consider estimating a density f in $\mathbb{R}^d$

- kernel function K: symmetric density (e.g., density of $\mathcal{N}(0, I_d)$)
- KDE: for a symmetric, PSD bandwidth matrix $\boldsymbol{H} \in \mathbb{R}^{d \times d}$

$$\hat{f}(\boldsymbol{x}) = \frac{1}{n} \sum_{i=1}^{n} \det(\boldsymbol{H})^{-1/2} K\big(\boldsymbol{H}^{-1/2}(\boldsymbol{x} - \boldsymbol{x}_i)\big)$$

- bandwidth selection: Silverman's rule of thumb

$$H_{i,i} = \Big(\frac{4}{n(d+2)}\Big)^{2/(d+4)} \sigma_i \quad (1 \le i \le d), \qquad H_{i,j} = 0 \quad (i \neq j).$$

  where σ_i^2 is the variance of the i-th variable.

- suffers from curse of dimensionality (error exponential in d)

Kernel Density Classifier

Bayes optimal classifier:

$$\mathbb{P}(Y = k \mid X = x) = \frac{\mathbb{P}(X = x \mid Y = k)\,\mathbb{P}(Y = k)}{\sum_{k'=1}^{K} \mathbb{P}(X = x \mid Y = k')\,\mathbb{P}(Y = k')}$$

- Construct a KDE $\widehat{f}_k(x)$ for the conditional density $\mathbb{P}(X = x \mid Y = k)$ using data $\{x_i : y_i = k\}$ for each class $k \in \{1, \ldots, K\}$,
- Estimate class priors $\mathbb{P}(y = k)$ with empirical frequency $\widehat{\pi}_k = n_k/n$
- Kernel density classifier: for any input x, return

$$\underset{1 \le k \le K}{\arg\max}\ \widehat{\mathbb{P}}(Y = k \mid X = x) := \frac{\widehat{\pi}_k \widehat{f}_k(x)}{\sum_{k'=1}^{K} \widehat{\pi}_{k'} \widehat{f}_{k'}(x)}$$

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- Issue: curse of dimensionality

Naive Bayes Classifier

- The Naive Bayes model assumes that given a class $Y = k$, the features $X_1, \ldots, X_d$ are conditionally independent.
- The class-conditional density $f_k(x) \equiv \mathbb{P}(X = x \mid Y = k)$ is given by:

$$f_k(x) = \prod_{j=1}^{d} f_{k,j}(x_j) \quad \text{where} \quad x = (x_1, \ldots, x_d)$$

  where $f_{k,j}(X_k)$ is the marginal density of X_j conditional on $Y = k$
- Naive Bayes classifier: for any input x, return

$$\arg\max_{1 \le k \le K} \widehat{\mathbb{P}}(Y = k \mid X = x) := \frac{\widehat{\pi}_k \widehat{f}_k(x)}{\sum_{k'=1}^{K} \widehat{\pi}_{k'} \widehat{f}_{k'}(x)}$$

  where $\widehat{f}_k(x) = \prod_{j=1}^{d} \widehat{f}_{k,j}(x_j)$.
- The estimate $\widehat{f}_{k,j}$ for class-conditional marginal densities $f_{k,j}$ can be computed using e.g., one-dimensional KDE or histogram

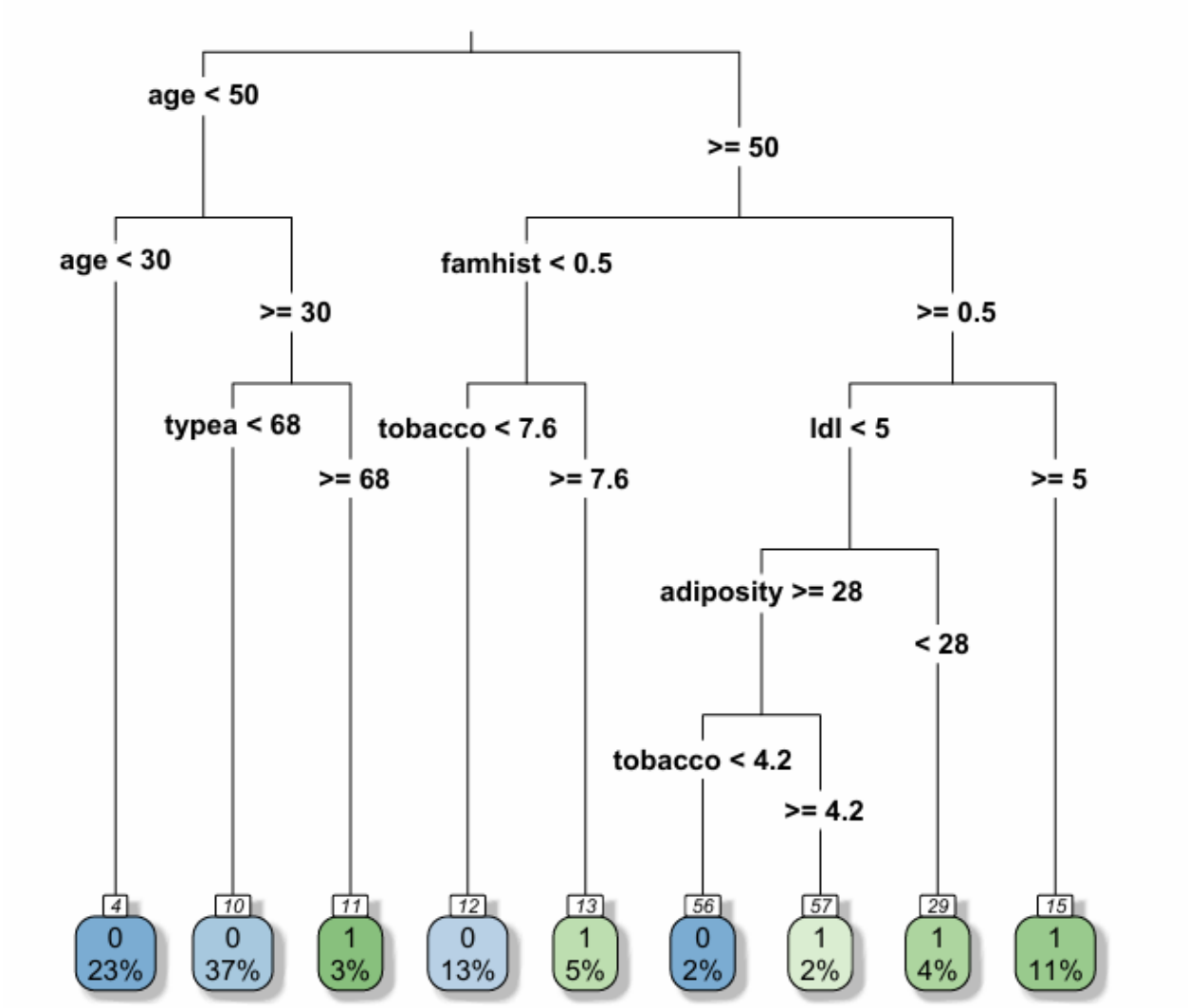
Discussions

- Naive Bayes works well in high-dimensional spaces and with small datasets, despite the independence assumption often being violated.
- **Advantages:**
  - Simple and fast
  - Avoids curse of dimensionality
  - Robust to irrelevant features
- **Disadvantages:**
  - Assumption of feature independence might be unrealistic

Tree-based methods

Classification tree

South African heart disease data: "0"="Yes, Disease", "1"="No"



Classification tree

Setup: $\mathcal{X} = \mathbb{R}^d$, $\mathcal{Y} = \{1, \ldots, K\}$, training data $(X_1, Y_1), \ldots, (X_n, Y_n)$

Idea: grow a tree to recursively partition the feature space into a set of rectangles, and do a simple majority vote in each rectangle

- Each node represents a rectangle in the feature space. The root node is the feature space $\mathcal{X} = \mathbb{R}^d$
- Each node is either a leaf (no children) or a parent (has two children)
- The left and right children comes from a partition of their parent node
- Suppose we have a collection of final partitioned regions associated with the leaves at the bottom of the tree, denoted by $R_1, \ldots, R_M$
- For any input x, suppose that $x \in R_j$, then this classification tree returns

$$\widehat{f}(x) = \arg\max_{k \in \mathcal{Y}} \sum_{X_i \in R_j} \mathbb{1}\{Y_i = k\}$$

  i.e., the predicted label is the majority in the region R_j

How to grow a classification tree?

In order to grow a classification tree, we need to ask:

1. How to split each parent node?
2. How large should we grow the tree?

For the first question: **minimizing impurity**

- Suppose that the parent node is associated with a rectangle R
- Choose a covariate X_j and a split point t that minimizes the impurity
- Let the rectangles associated with its left and right children be

$$R_1(j,t) = \{X \in R : X_j \leq t\} \quad \text{and} \quad R_2(j,t) = \{X \in R : X_j > t\},$$

For the second question: **set some stopping criteria**.

- For example, we may fix some number n_0, and we might stop partition a node when its associated rectangle has fewer than n_0 training data points.

Impurity function

Let R be the node to be split into two regions. We choose

$$\arg\min_{j,t} \underbrace{\frac{|R_1(j,t)|}{|R|}\gamma(R_1(j,t)) + \frac{|R_2(j,t)|}{|R|}\gamma(R_2(j,t))}_{\text{impurity function}},$$

- Here $\gamma(R)$ measures the "variance" of the labels of data in R: we want

$$\{Y_i : X_i \in R\} \quad \text{to have low variability}$$

- For any given rectangle R, let

$$p_k = \frac{1}{|R|}\sum_{X_i \in R} \mathbb{1}\{Y_i = k\}, \quad 1 \le k \le K.$$

  Two common choice of the function $\gamma(\cdot)$:

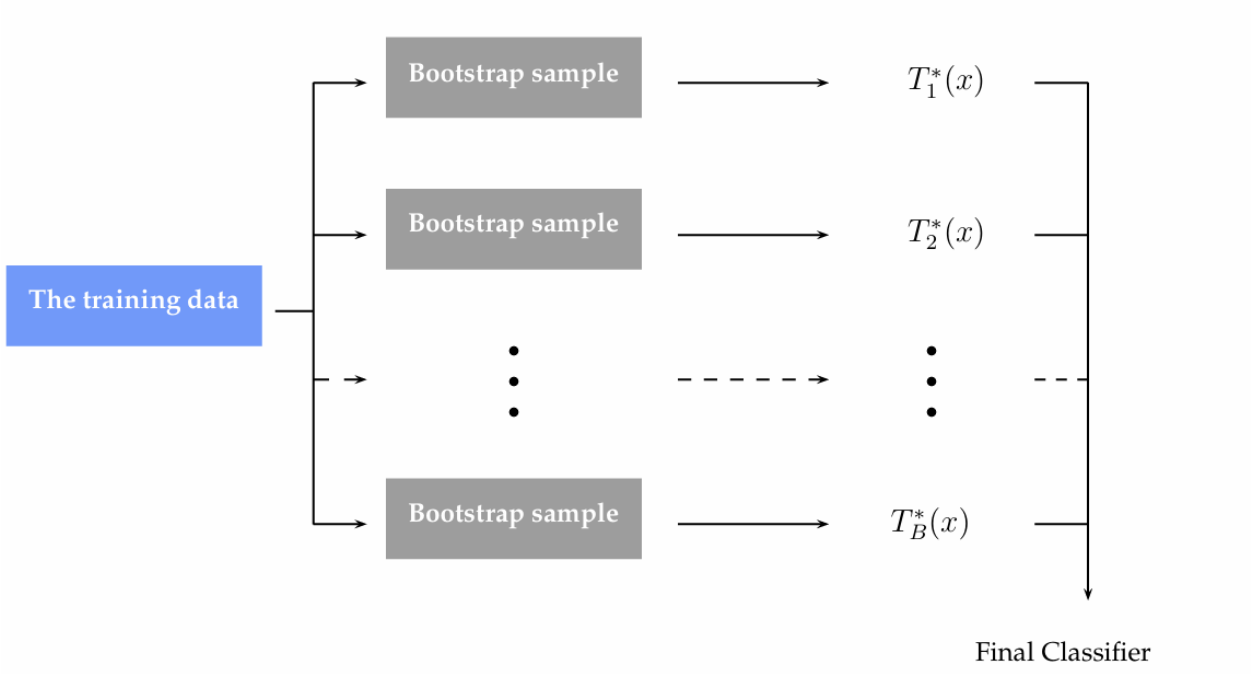
  - **Gini index**: $\gamma(R) = \sum_k p_k(1-p_k)$
  - **Cross entropy**: $\gamma(R) = -\sum_k p_k \log p_k$

Insights

- **advantage:** the tree structure provides great interpretability
  - for example, it allows reasoning about the cause of diseases
- **disadvantage:** instability due to the use of *greedy* search:
  - splitting process is greedy
  - small changes in the training data can lead to significantly different tree structures
- **Solutions:**
  - Regularization: controlling tree growth parameters
  - Pruning: removing branches that do not provide significant predictive power
  - Ensemble Methods: use bagging to create a random forest

Bootstrap aggregating (Bagging)

- Training data $Z_n = \{(X_i, Y_i), 1 \le i \le n\}$
- Bootstrap sample $Z^{(*b)} = \{(X_i^{(*b)}, Y_i^{(*b)}), 1 \le i \le n\}$: sample n data points randomly from Z_n with replacement
- Apply the learning algorithm to the bootstrap sample for B times, and produce outcomes $\widehat{f}_b$
- Majority vote: $\widehat{f}^{\text{bagging}}(x) = \arg\max_{k \in \mathcal{Y}} \sum_{b=1}^{B} \mathbb{1}\{\widehat{f}_b(x) = k\}$



Insights

- Trees generated in bagging are identically distributed (not independent!)
- Bias of bagged tress is the same as the individual tree
- **Pro**: Reduce the variance, so good for high-variance, low-bias procedures, like trees.
- **Heuristics**: Suppose we have B identically distributed random variables with variance σ^2 and positive pairwise correlation ρ, then their average has variance of

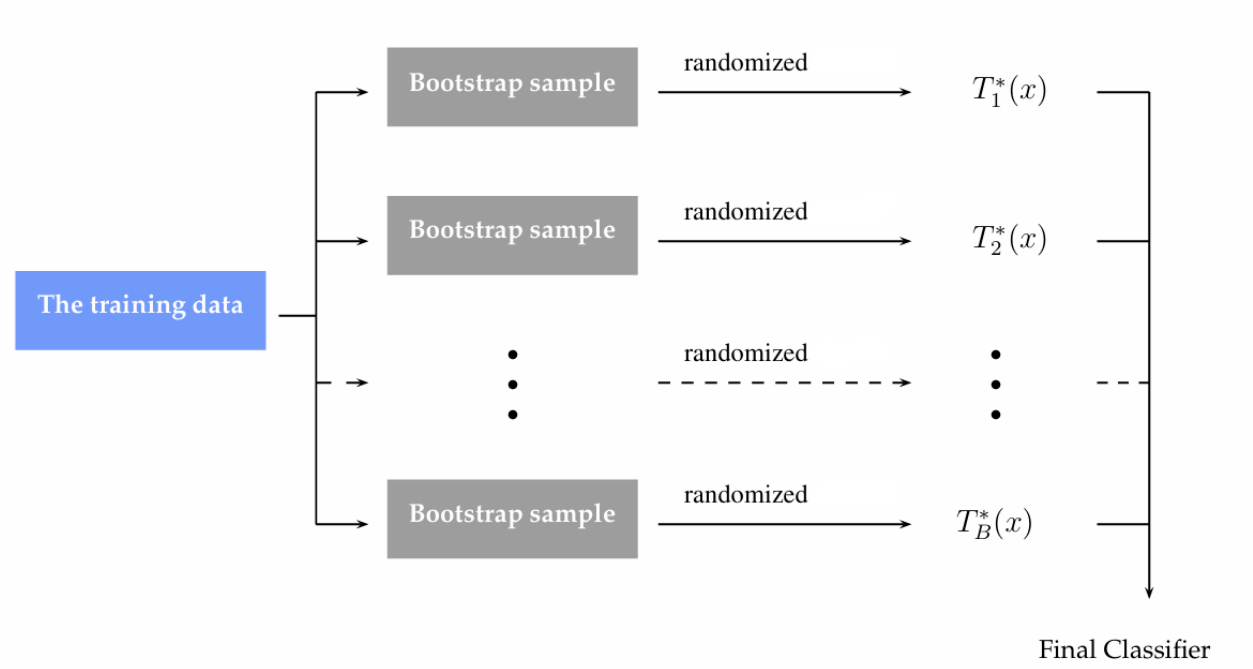
$$\rho\sigma^2 + \frac{1-\rho}{B}\sigma^2$$

- Increasing B does not reduce the first term

— Random Forest!

Random forests

- **Key idea**: use random dropout to decorrelate bootstrapped trees
- When growing a tree on a bootstrapped sample, before each split of the node, select $m \ll d$ variables at random as candidates to split
- Typical values for m is $\sqrt{d}$.
- Majority vote: $\widehat{f}^{\mathsf{RF}}(x) = \arg\max_{k \in \mathcal{Y}} \sum_{b=1}^{B} \mathbb{1}\{\widetilde{f}_b(x) = k\}$



How to remove bias: Boosting

- Setup: $\mathcal{X} = \mathbb{R}^d$, $\mathcal{Y} = \{\pm 1\}$
- Weak classifier: error rate only slightly better than random guess
- Key idea: sequentially apply weak classification algorithm to repeatedly modified versions of the data to produce a sequence of weak classifiers
  - assign unequal weights to training data points

    — *possible for trees*
  - sequentially find a committee of weak classifiers $\{\widehat{f}_m\}_{m=1}^M$
  - produce the final prediction through a weighted majority vote

$$\widehat{f}(x) := \text{sign}(\sum_{m=1}^{M} \alpha_m \widehat{f}_m(x))$$

AdaBoost

Initialization: set the weights $w_i = 1/n$ for $1 \leq i \leq n$.
For $m = 1, \ldots, M$:

- Fit a weak classifier $\widehat{f}_m(x)$ using training data with weights $\omega_1, \ldots, \omega_n$
- Compute the weighted misclassification error:

$$\mathsf{err}^{(m)} = \frac{\sum_{i=1}^n w_i \, \mathbb{1}\{Y_i \neq \widehat{f}_m(X_i)\}}{\sum_{i=1}^n w_i}.$$

- Compute:

$$\alpha_m = \log\Big(\frac{1 - \mathsf{err}^{(m)}}{\mathsf{err}^{(m)}}\Big).$$

- Update the weights by:

$$w_i \leftarrow w_i \cdot \exp\big(\alpha_m \cdot \mathbb{1}\{Y_i \neq \widehat{f}_m(X_i)\}\big), \quad i = 1, 2, \ldots, n.$$

Output: $\widehat{f}(x) = \mathsf{sign}\big(\sum_{m=1}^M \alpha_m \widehat{f}_m(x)\big)$.

AdaBoost: insights

Key idea: in the weight update step

$$w_i \leftarrow w_i \cdot \exp\left(\alpha_m \cdot \mathbb{1}\{Y_i \neq \widehat{f}_m(X_i)\}\right), \quad i = 1, 2, \ldots, n.$$

- For incorrectly classified data points, their weights get inflated by e^{α_m}
- Note that $\alpha_m > 0$ should always hold
- This re-weighting encourages the next classifier to focus more on the misclassified data points

Discussion: three main approaches to classification

Three main approaches

Bayes optimal classifier: for any $x \in \mathcal{X}$, output

$$f^\star(x) \coloneqq \arg\max_{y \in \mathcal{Y}} \mathbb{P}(Y = y \mid X = x)$$

minimizes the Bayes risk $R(f) = \mathbb{P}(f(X) \neq Y)$

Three main approaches

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minimizes the Bayes risk $R(f) = \mathbb{P}(f(X) \neq Y)$

- Plug-in approach: model data distribution ρ, then estimate densities

$$\mathbb{P}(Y = y \mid X = x) = \frac{\mathbb{P}(X = x \mid Y = y)\,\mathbb{P}(Y = y)}{\sum_{y' \in \mathcal{Y}} \mathbb{P}(X = x \mid Y = y')\,\mathbb{P}(Y = y')}$$

Three main approaches

Bayes optimal classifier: for any $x \in \mathcal{X}$, output

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- Plug-in approach: model data distribution ρ, then estimate densities

$$\mathbb{P}(Y = y \mid X = x) = \frac{\widehat{\mathbb{P}}(X = x \mid Y = y)\, \widehat{\mathbb{P}}(Y = y)}{\sum_{y' \in \mathcal{Y}} \widehat{\mathbb{P}}(X = x \mid Y = y')\, \widehat{\mathbb{P}}(Y = y')}$$

  Example: LDA, QDA, Kernel density classifier

Three main approaches

Bayes optimal classifier: for any $x \in \mathcal{X}$, output

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minimizes the Bayes risk $R(f) = \mathbb{P}(f(X) \neq Y)$

- Plug-in approach
- Regression: modeling and estimating each

$$r_k(x) \coloneqq \mathbb{P}(Y = k \mid X = x) \quad \text{for} \quad k = 1, \ldots, K$$

Three main approaches

Bayes optimal classifier: for any $x \in \mathcal{X}$, output

$$f^\star(x) \coloneqq \arg\max_{y \in \mathcal{Y}} \mathbb{P}(Y = y \mid X = x)$$

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- Plug-in approach
- Regression: modeling and estimating each

$$r_k(x) \coloneqq \mathbb{P}(Y = k \mid X = x) \quad \text{for} \quad k = 1, \ldots, K$$

  Example: logistic regression

Three main approaches

Bayes optimal classifier: for any $x \in \mathcal{X}$, output

$$f^\star(x) \coloneqq \arg\max_{y \in \mathcal{Y}} \mathbb{P}(Y = y \mid X = x)$$

minimizes the Bayes risk $R(f) = \mathbb{P}(f(X) \neq Y)$

- Plug-in approach
- Regression
- Empirical risk minimization: choose a set of classifiers $\mathcal{F}$ and find $\widehat{f} \in \mathcal{F}$ that minimizes the "empirical risk":

$$R_n(f) \coloneqq \frac{1}{n} \sum_{i=1}^{n} \mathbb{1}\{f(X_i) \neq Y_i\}$$

  Intuition: when n is large, $R_n(f) \approx R(f)$ by LLN

Three main approaches

Bayes optimal classifier: for any $x \in \mathcal{X}$, output

$$f^\star(x) := \arg\max_{y \in \mathcal{Y}} \mathbb{P}(Y = y \mid X = x)$$

minimizes the Bayes risk $R(f) = \mathbb{P}(f(X) \neq Y)$

- Plug-in approach
- Regression
- Empirical risk minimization
- Other approaches: SVM, tree-based methods...

ERM: advantages

$$\widehat{f}_n = \arg\min_{f \in \mathcal{F}} \frac{1}{n} \sum_{i=1}^{n} \mathbb{1}\{f(X_i) \neq Y_i\} =: R_n(f)$$

- a straightforward method based on simple heuristics

  — *robustness!*

- can be easily generalized to other loss $\ell(\cdot, \cdot)$ by considering

  $$R_n(f) := \frac{1}{n} \sum_{i=1}^{n} \ell(f(X_i), Y_i)$$

  if the ultimate goal is to minimize $R_\ell(f) = \mathbb{E}[\ell(f(X), Y)]$. For example, in binary classification (i.e., $\mathcal{Y} = \{0, 1\}$)

  - Hinge loss $\ell(f(x), y) = \max\{0, 1 - yf(x)\}$
  - Logistic loss $\ell(f(x), y) = \log(1 + \exp(-yf(x)))$

  — *Logistic regression can also be viewed as ERM!*

ERM: disadvantages

$$\widehat{f}_n = \arg\min_{f \in \mathcal{F}} \frac{1}{n} \sum_{i=1}^{n} \mathbb{1}\{f(X_i) \neq Y_i\} =: R_n(f)$$

- Not easy to compute (due to nonsmoothness of the indicator function)
- Solution: in binary classification (i.e., $\mathcal{Y} = \{0, 1\}$), consider using hinge loss or logistic loss $\ell(\cdot)$

  $$R_n(f) := \frac{1}{n} \sum_{i=1}^{n} \ell(f(X_i), Y_i)$$

  and relax $f : \mathbb{R}^d \to \mathbb{R}$, and finally output $\text{sign}(2(f(x) - 1))$
- Here we will only focus on the standard ERM

ERM: error decomposition

$$\widehat{f}_n = \arg\min_{f \in \mathcal{F}} \frac{1}{n} \sum_{i=1}^{n} \mathbb{1}\{f(X_i) \neq Y_i\} =: R_n(f)$$

- We want to control the excess risk

$$R(\widehat{f}_n) - R(f^\star) = \underbrace{R(\widehat{f}_n) - \min_{f \in \mathcal{F}} R(f)}_{\geq 0, \text{ statistical error}} + \underbrace{\min_{f \in \mathcal{F}} R(f) - R(f^\star)}_{\geq 0, \text{ approximation error}}$$

- approximation error: becomes smaller when choosing larger $\mathcal{F}$

  — *becomes 0 when $f^\star \in \mathcal{F}$*

- statistical error: becomes smaller when n becomes larger, and when choosing smaller $\mathcal{F}$ (why?)

- **trade-off between fit and complexity**

- In this course, we will focus on understanding statistical error with a given $\mathcal{F}$ that includes $f^\star$ (so that approximation error $= 0$)

Excess risk via uniform deviations

$$\widehat{f}_n = \arg\min_{f \in \mathcal{F}} \frac{1}{n} \sum_{i=1}^{n} \mathbb{1}\{f(X_i) \neq Y_i\} =: R_n(f)$$

Theorem 2.8

The excess risk is upper bounded by

$$R(\widehat{f}_n) - R(f^\star) \leq 2 \sup_{f \in \mathcal{F}} |R_n(f) - R(f)|$$

Implications:

- For a given f, we know that $R_n(f) \to R(f)$ at a rate $O(1/\sqrt{n})$ by CLT

$$\sqrt{n}\big(R_n(f) - R(f)\big) \xrightarrow{\mathrm{d}} \mathcal{N}(0, \mathsf{var}(\mathbb{1}\{f(X) \neq Y\}))$$

- But what about the uniform convergence of $\sup_{f \in \mathcal{F}} |R_n(f) - R(f)|$?

Concentration inequalities and uniform convergence

Why concentration inequalities?

Consider i.i.d. variables $X_1, \ldots, X_n$ with $\mathbb{E}[X_i] = \mu$ and $\text{var}(X_i) = \sigma^2$

- Central limit theorem (CLT):

$$\sqrt{n}\Big(\frac{1}{n}\sum_{i=1}^{n} X_i - \mu\Big) \xrightarrow{\text{d}} \mathcal{N}(0, \sigma^2)$$

  tells us that the sample average concentrates around μ, and the deviation scales like $\sigma/\sqrt{n}$ as $n \to \infty$

- But this does not say anything useful when n is finite

- We want some non-asymptotic statement like:

$$\mathbb{P}\Big(\Big|\frac{1}{n}\sum_{i=1}^{n} X_i - \mu\Big| \geq \varepsilon(n, \delta)\Big) \leq \delta$$

  holds for any $\delta > 0$, where $\varepsilon(n, \delta) > 0$ is some quantity that depends on the sample size n and the exceptional probability δ

A simple case with i.i.d. Gaussian

Suppose that $X_1, \ldots, X_n \overset{\text{i.i.d.}}{\sim} \mathcal{N}(\mu, \sigma^2)$, then we have

$$\frac{1}{n}\sum_{i=1}^{n} X_i - \mu \sim \mathcal{N}\Big(0, \frac{\sigma^2}{n}\Big)$$

Theorem 2.9

For $G \sim \mathcal{N}(0,1)$ and any $t > 0$, we have

$$\Big(\frac{1}{t} - \frac{1}{t^3}\Big)\frac{1}{\sqrt{2\pi}} e^{-t^2/2} \leq \mathbb{P}(G \geq t) \leq \frac{1}{t}\frac{1}{\sqrt{2\pi}} e^{-t^2/2}.$$

As a result,

$$\mathbb{P}\Big(\Big|\frac{1}{n}\sum_{i=1}^{n} X_i - \mu\Big| \geq t\Big) \leq \frac{2\sigma}{\sqrt{n}t}\frac{1}{\sqrt{2\pi}} \exp\Big(-\frac{nt^2}{2\sigma^2}\Big)$$

A simple case with i.i.d. Gaussian

Suppose that $X_1, \ldots, X_n \overset{\text{i.i.d.}}{\sim} \mathcal{N}(\mu, \sigma^2)$, then we have

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Question: how can we extend these to more general distributions?

From Gaussian to sub-Gaussian

- Question: can we generalize these results to other random variables?
- Idea: consider other random variables with similar tail probability
- From Theorem 3.2, we know that for $G \sim \mathcal{N}(0, \sigma^2)$,

$$\mathbb{P}(|G| \geq t) \lesssim e^{-t^2/\sigma^2} \quad \text{for all } t \geq 0$$

- We may consider random variables satisfy this type of tail properties

— *sub-Gaussian*

Sub-Gaussian properties

Let X be a random variable, then the following properties are equivalent:

1. The tails of X satisfy
$$\mathbb{P}(|X| \geq t) \leq 2\exp\left(-t^2/K_1^2\right) \quad \text{for all } t \geq 0$$
2. The moments of X satisfy
$$\|X\|_{L^p} := (\mathbb{E}[|X|^p])^{1/p} \leq K_2\sqrt{p} \quad \text{for all } p \geq 1$$
3. The moment generating function (MGF) of X^2 satisfies
$$\mathbb{E}\big[\exp(\lambda^2 X^2)\big] \leq \exp(K_3^2\lambda^2) \quad \text{for all } \lambda \text{ such that } |\lambda| \leq 1/K_3$$
4. The MGF of X^2 is bounded at some point, namely
$$\mathbb{E}\big[\exp\left(X^2/K_4^2\right)\big] \leq 2.$$
5. If $\mathbb{E}X = 0$, then the MGF of X satisfies
$$\mathbb{E}\big[\exp(\lambda X)\big] \leq \exp(K_5^2\lambda^2) \quad \text{for all } \lambda \in \mathbb{R}.$$

where $K_1, \ldots, K_5 > 0$ may differ by at most a multiplicative constant factor

Sub-Gaussian distributions: definition

- If X satisfies one of properties 1-4, it is a *sub-Gaussian random variable*.
- The *sub-Gaussian norm* of X, denoted $\|X\|_{\psi_2}$, is defined to be the smallest K_4 in property 4. In other words, we define

$$\|X\|_{\psi_2} = \inf\left\{t > 0 : \mathbb{E}\exp\left(X^2/t^2\right) \leq 2\right\}.$$

— can also be defined using K_1, K_2 or K_3

- Properties: there exists some absolute constants $c, C > 0$ such that
  - $P(|X| \geq t) \leq 2\exp\left(-ct^2/\|X\|_{\psi_2}^2\right)$
  - $\|X\|_{L^p} \leq C\|X\|_{\psi_2}\sqrt{p}$
  - $\mathbb{E}\exp\left(X^2/\|X\|_{\psi_2}^2\right) \leq 2$
  - if $\mathbb{E}[X] = 0$, then $\mathbb{E}\exp(\lambda X) \leq \exp(C\lambda^2\|X\|_{\psi_2}^2)$

Sub-Gaussian distributions: examples

- **Gaussian:** if $X \sim \mathcal{N}(0, \sigma^2)$, then X is sub-Gaussian with

$$\|X\|_{\psi_2} \leq C\sigma$$

  for some universal constant $C = 2\sqrt{2/3}$.

- **Bounded:** any bounded random variable X is sub-Gaussian with

$$\|X\|_{\psi_2} \leq C\|X\|_\infty$$

  for some universal constant $C = 1/\sqrt{\log 2}$.

Sub-Gaussian norm can be viewed as a characterization of "magnitude" for light tail distributions.

Centering and independent sums

Theorem 2.10

- *If X is sub-Gaussian, then $X - \mathbb{E}[X]$ is sub-Gaussian with*

$$\|X - \mathbb{E}[X]\|_{\psi_2} \leq C\|X\|_{\psi_2}$$

where C is an absolute constant.

- *Let $X_1, \ldots, X_N$ be independent, mean zero, sub-Gaussian random variables. Then the sum $S_N = \sum_{i=1}^N X_i$ is also sub-Gaussian, and its sub-Gaussian norm satisfies*

$$\|S_N\|_{\psi_2}^2 \leq C \sum_{i=1}^N \|X_i\|_{\psi_2}^2,$$

where C is an absolute constant.

Analog:

- If $X_1, \ldots, X_n$ are i.i.d. $\mathcal{N}(0, \sigma^2)$, then $S_N \sim \mathcal{N}(0, N\sigma^2)$
- If $X_1, \ldots, X_n$ are independent with $\|X_i\|_{\psi_2} \leq \sigma$, then $\|S_N\|_{\psi_2} \lesssim \sqrt{N}\sigma$

Hoeffding's inequality

Theorem 2.11 (Hoeffding's Inequality)

Let $X_1, \ldots, X_N$ be independent, mean-zero, sub-Gaussian random variables. Then, for any $t \geq 0$, we have:

$$\mathbb{P}\left(\left|\sum_{i=1}^{N} X_i\right| \geq t\right) \leq 2\exp\left(-\frac{ct^2}{\sum_{i=1}^{N} \|X_i\|_{\psi_2}^2}\right),$$

where c is an absolute constant.

Implications

- **General Hoeffding:** under the setup of Theorem 3.4, consider any vector $\boldsymbol{a} = (a_1, \ldots, a_n) \in \mathbb{R}^n$, we have

$$\mathbb{P}\left(\left|\sum_{i=1}^{N} a_i X_i\right| \geq t\right) \leq 2\exp\left(-\frac{ct^2}{K^2\|\boldsymbol{a}\|_2^2}\right),$$

  where $K := \max \|X_i\|_{\psi_2}$.

- **Example:** suppose that $X_i \sim$ Bernoulli(p_i) for $1 \leq i \leq n$, then

$$\mathbb{P}\left(\left|\sum_{i=1}^{N}(X_i - p_i)\right| \geq t\right) \leq 2\exp\left(-\frac{ct^2}{N}\right),$$

  A sharper result for binomial concentration: Chernoff's inequality (HW)

Back to ERM: finite $\mathcal{F}$

$$\widehat{f}_n = \arg\min_{f \in \mathcal{F}} \frac{1}{n} \sum_{i=1}^{n} \mathbb{1}\{f(X_i) \neq Y_i\} =: R_n(f)$$

Theorem 2.12

Suppose that $\mathcal{F}$ is a finite set. Then with probability exceeding $1 - \delta$, the excess risk of ERM is upper bounded by

$$R(\widehat{f}_n) - R(f^\star) \leq C\sqrt{\frac{\log(|\mathcal{F}|/\delta)}{n}}.$$

for some universal constant $C > 0$.

- Key proof idea: **union bound argument**
- What if $\mathcal{F}$ is not finite (e.g., the set of linear classifiers)?

— *use VC dimension!*

Back to ERM: finite $\mathcal{F}$

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for some universal constant $C > 0$.

- Key proof idea: **union bound argument**
- What if $\mathcal{F}$ is not finite (e.g., the set of linear classifiers)?

— *use VC dimension!*

- But before going into that, let's first warm up with something simpler

ℓ_2 norm of a sub-Gaussian random vector

- Consider a random vector $\boldsymbol{x} = (X_1, \ldots, X_d)$, where $X_1, \ldots, X_d$ are independent random variables with $\mathbb{E}[X_i] = 0$ and $\|X_i\|_{\psi_2} \leq \sigma$
- Can we establish a non-asymptotic upper bound for $\|\boldsymbol{x}\|_2$?

ℓ_2 norm of a sub-Gaussian random vector

- Consider a random vector $\boldsymbol{x} = (X_1, \ldots, X_d)$, where $X_1, \ldots, X_d$ are independent random variables with $\mathbb{E}[X_i] = 0$ and $\|X_i\|_{\psi_2} \leq \sigma$
- Can we establish a non-asymptotic upper bound for $\|\boldsymbol{x}\|_2$?
- Solution 1: entrywise concentration and union bound

$$\mathbb{P}\big(\|\boldsymbol{x}\|_2 \leq C\sigma\sqrt{d\log(d/\delta)}\big) \geq 1 - \delta$$

for some universal constant $C > 0$

ℓ_2 norm of a sub-Gaussian random vector

- Consider a random vector $\boldsymbol{x} = (X_1, \ldots, X_d)$, where $X_1, \ldots, X_d$ are independent random variables with $\mathbb{E}[X_i] = 0$ and $\|X_i\|_{\psi_2} \leq \sigma$
- Can we establish a non-asymptotic upper bound for $\|\boldsymbol{x}\|_2$?
- Solution 1: entrywise concentration and union bound

$$\mathbb{P}\big(\|\boldsymbol{x}\|_2 \leq C\sigma\sqrt{d\log(d/\delta)}\big) \geq 1 - \delta$$

  for some universal constant $C > 0$

- Solution 2: uniform concentration using

$$\|\boldsymbol{x}\|_2 = \sup_{\boldsymbol{a} \in \mathcal{S}^{d-1}} \boldsymbol{a}^\top \boldsymbol{x}$$

  where $\mathcal{S}^{d-1} := \{\boldsymbol{x} \in \mathbb{R}^d : \|\boldsymbol{x}\|_2 = 1\}$ is the unit sphere in $\mathbb{R}^d$

  — *could this provide a better concentration bound?*

Operator norm of sub-Gaussian matrix

- Consider a random matrix $\boldsymbol{X} = (X_{i,j})_{1 \le i,j \le d}$ with independent entries that satisfies $\mathbb{E}[X_{i,j}] = 0$ and $\|X_{i,j}\|_{\psi_2} \le \sigma$
- Can we establish a non-asymptotic upper bound for $\|\boldsymbol{X}\|$?
- Operator norm:

$$\|\boldsymbol{X}\| = \sup_{\boldsymbol{a} \in \mathcal{S}^{d-1}} \|\boldsymbol{X}\boldsymbol{a}\|_2 = \sup_{\boldsymbol{a}, \boldsymbol{b} \in \mathcal{S}^{d-1}} \boldsymbol{a}^\top \boldsymbol{X} \boldsymbol{b}$$

where $\mathcal{S}^{d-1} := \{\boldsymbol{x} \in \mathbb{R}^d : \|\boldsymbol{x}\|_2 = 1\}$ is the unit sphere in $\mathbb{R}^d$

A framework for uniform concentration

- **Goal:** upper bounding $\sup_{\boldsymbol{a} \in \mathcal{S}^{d-1}} \boldsymbol{a}^\top \boldsymbol{x}$
- **Step 1: pointwise concentration.** For any fixed $\boldsymbol{a} \in \mathcal{S}^{d-1}$, we can use Hoeffding's inequality to get

  $$\mathbb{P}\big(|\boldsymbol{a}^\top \boldsymbol{x}| \leq C\sigma\sqrt{\log(1/\delta)}\big) \geq 1 - \delta$$

  for some universal constant $C > 0$
- **Difficulty:** the unit sphere $\mathcal{S}^{d-1}$ is not a finite set, union bound argument does not work
- **Idea:** find a finite subset $\mathcal{N}$ of $\mathcal{S}^{d-1}$ that is *fine* enough, such that

  $$\sup_{\boldsymbol{a} \in \mathcal{S}^{d-1}} \boldsymbol{a}^\top \boldsymbol{x} \overset{?}{\lesssim} \sup_{\boldsymbol{a} \in \mathcal{N}} \boldsymbol{a}^\top \boldsymbol{x} \leq C\sigma\sqrt{\log(|\mathcal{N}|/\delta)}$$

  with probability at least $1 - \delta$

Epsilon net

- Let (T, d) be a metric space. Consider a subset $K \subset T$ and let $\varepsilon > 0$.

  — *e.g., consider $T = \mathbb{R}^d$, $d(\cdot, \cdot)$ is Euclidean distance, $K = \mathcal{S}^{d-1}$*

- A subset $N \subseteq K$ is called an ε-net of K if every point in K is within distance ε of some point of N, i.e.,

$$\forall\, x \in K, \quad \exists\, x_0 \in N \quad \text{s.t.} \quad d(x, x_0) \leq \varepsilon.$$

Theorem 2.13

Let $\mathcal{N}_\varepsilon$ be an ε-net of $\mathcal{S}^{d-1}$. If $\varepsilon < 1$, then for any $\boldsymbol{x} \in \mathbb{R}^d$,

$$\sup_{\boldsymbol{a} \in \mathcal{N}_\varepsilon} \boldsymbol{a}^\top \boldsymbol{x} \leq \sup_{\boldsymbol{a} \in \mathcal{S}^{d-1}} \boldsymbol{a}^\top \boldsymbol{x} \leq \frac{1}{1-\varepsilon} \sup_{\boldsymbol{a} \in \mathcal{N}_\varepsilon} \boldsymbol{a}^\top \boldsymbol{x},$$

and if $\varepsilon < 1/2$, then for any $\boldsymbol{X} \in \mathbb{R}^{d \times d}$,

$$\sup_{\boldsymbol{a}, \boldsymbol{b} \in \mathcal{N}_\varepsilon} \boldsymbol{a}^\top \boldsymbol{X} \boldsymbol{b} \leq \sup_{\boldsymbol{a}, \boldsymbol{b} \in \mathcal{S}^{d-1}} \boldsymbol{a}^\top \boldsymbol{X} \boldsymbol{b} \leq \frac{1}{1-2\varepsilon} \sup_{\boldsymbol{a}, \boldsymbol{b} \in \mathcal{N}_\varepsilon} \boldsymbol{a}^\top \boldsymbol{X} \boldsymbol{b}.$$

The covering number

Covering number: the smallest possible cardinality of an ε-net of K, denoted by $\mathcal{N}(K, \varepsilon)$

Theorem 2.14

The covering number of $\mathcal{S}^{d-1}$ is upper bounded by

$$\mathcal{N}(\mathcal{S}^{d-1}, \varepsilon) \leq \left(\frac{2}{\varepsilon} + 1\right)^d$$

ℓ_2 norm of sub-Gaussian random vector

- **Goal:** upper bounding $\sup_{\boldsymbol{a} \in \mathcal{S}^{d-1}} \boldsymbol{a}^\top \boldsymbol{x}$
- **Step 1: pointwise concentration.** For any fixed $\boldsymbol{a} \in \mathcal{S}^{d-1}$, we can use Hoeffding's inequality to get

$$\mathbb{P}\big(|\boldsymbol{a}^\top \boldsymbol{x}| \leq C_1 \sigma \sqrt{\log(1/\delta)}\big) \geq 1 - \delta$$

  for some universal constant $C_1 > 0$

ℓ_2 norm of sub-Gaussian random vector

- **Goal:** upper bounding $\sup_{\boldsymbol{a} \in \mathcal{S}^{d-1}} \boldsymbol{a}^\top \boldsymbol{x}$
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$$\mathbb{P}\big(|\boldsymbol{a}^\top \boldsymbol{x}| \le C_1 \sigma \sqrt{\log(1/\delta)}\big) \ge 1 - \delta$$

  for some universal constant $C_1 > 0$
- **Step 2: uniform concentration over an $1/2$-net.** Let $\mathcal{N}_{1/2}$ be the smallest $1/2$-net of $\mathcal{S}^{d-1}$. By union bound argument and Theorem 3.7,

$$\mathbb{P}\Big(\sup_{\boldsymbol{a} \in \mathcal{N}_{1/2}} |\boldsymbol{a}^\top \boldsymbol{x}| \le C_2 \sigma \sqrt{d \log(1/\delta)}\Big) \ge 1 - \delta$$

  for some universal constant $C_2 > 0$

ℓ_2 norm of sub-Gaussian random vector

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  for some universal constant $C_2 > 0$
- **Step 3: approximation.** By Theorem 3.6,

$$\mathbb{P}\big(\|\boldsymbol{x}\|_2 \le C_3 \sigma \sqrt{d \log(1/\delta)}\big) \ge 1 - \delta$$

  for some universal constant $C_3 > 0$

Operator norm of sub-Gaussian random matrix

- **Goal:** upper bounding $\sup_{\boldsymbol{a},\boldsymbol{b}\in\mathcal{S}^{d-1}} \boldsymbol{a}^\top \boldsymbol{X}\boldsymbol{b}$
- **Step 1: pointwise concentration.** For any fixed $\boldsymbol{a},\boldsymbol{b}\in\mathcal{S}^{d-1}$, we can use Hoeffding's inequality to get

$$\mathbb{P}\big(|\boldsymbol{a}^\top \boldsymbol{X}\boldsymbol{b}| \le C_1\sigma\sqrt{\log(1/\delta)}\big) \ge 1-\delta$$

for some universal constant $C_1 > 0$

Operator norm of sub-Gaussian random matrix

- **Goal:** upper bounding $\sup_{\boldsymbol{a},\boldsymbol{b}\in\mathcal{S}^{d-1}} \boldsymbol{a}^\top \boldsymbol{X}\boldsymbol{b}$
- **Step 1: pointwise concentration.** For any fixed $\boldsymbol{a},\boldsymbol{b}\in\mathcal{S}^{d-1}$, we can use Hoeffding's inequality to get

$$\mathbb{P}\big(|\boldsymbol{a}^\top \boldsymbol{X}\boldsymbol{b}| \le C_1\sigma\sqrt{\log(1/\delta)}\big) \ge 1-\delta$$

  for some universal constant $C_1 > 0$
- **Step 2: uniform concentration over an $1/4$-net.** Let $\mathcal{N}_{1/4}$ be the smallest $1/4$-net of $\mathcal{S}^{d-1}$. By union bound argument and Theorem 3.7,

$$\mathbb{P}\Big(\sup_{\boldsymbol{a},\boldsymbol{b}\in\mathcal{N}_{1/4}} |\boldsymbol{a}^\top \boldsymbol{X}\boldsymbol{b}| \le C_2\sigma\sqrt{d\log(1/\delta)}\Big) \ge 1-\delta$$

  for some universal constant $C_2 > 0$

Operator norm of sub-Gaussian random matrix

- **Goal:** upper bounding $\sup_{\boldsymbol{a},\boldsymbol{b}\in\mathcal{S}^{d-1}} \boldsymbol{a}^\top \boldsymbol{X}\boldsymbol{b}$
- **Step 1: pointwise concentration.** For any fixed $\boldsymbol{a},\boldsymbol{b}\in\mathcal{S}^{d-1}$, we can use Hoeffding's inequality to get

$$\mathbb{P}\big(|\boldsymbol{a}^\top \boldsymbol{X}\boldsymbol{b}| \le C_1\sigma\sqrt{\log(1/\delta)}\big) \ge 1-\delta$$

  for some universal constant $C_1 > 0$
- **Step 2: uniform concentration over an $1/4$-net.** Let $\mathcal{N}_{1/4}$ be the smallest $1/4$-net of $\mathcal{S}^{d-1}$. By union bound argument and Theorem 3.7,

$$\mathbb{P}\Big(\sup_{\boldsymbol{a},\boldsymbol{b}\in\mathcal{N}_{1/4}} |\boldsymbol{a}^\top \boldsymbol{X}\boldsymbol{b}| \le C_2\sigma\sqrt{d\log(1/\delta)}\Big) \ge 1-\delta$$

  for some universal constant $C_2 > 0$
- **Step 3: approximation.** By Theorem 3.6,

$$\mathbb{P}\big(\|\boldsymbol{X}\|_2 \le 2C_2\sigma\sqrt{d\log(1/\delta)}\big) \ge 1-\delta$$

  for some universal constant $C_3 > 0$

VC dimension

- Let $\mathcal{F}$ be a class of binary functions on the domain $\mathcal{X}$.
- **Shattering:** a set of points $\{x_1, \ldots, x_k\} \subseteq \mathcal{X}$ is shattered by $\mathcal{F}$ if for every possible labeling $\{0,1\}^k$, there exists a function $f \in \mathcal{F}$ that realizes the labeling.
- The **VC dimension** of $\mathcal{F}$, denoted $\text{VC}(\mathcal{F})$, is the largest integer k such that there exists a set of k points in $\mathcal{X}$ that can be *shattered* by $\mathcal{F}$.
- Examples:
  - When $\mathcal{X} = \mathbb{R}^2$, $\mathcal{F} =$ linear classifiers, we have $\text{vc}(\mathcal{F}) = 3$
  - In general, when $\mathcal{X} = \mathbb{R}^d$, $\mathcal{F} =$ linear classifiers, then $\text{vc}(\mathcal{F}) = d + 1$

Bounding excess risk via VC dimension

$$\widehat{f}_n = \arg\min_{f \in \mathcal{F}} \frac{1}{n} \sum_{i=1}^{n} \mathbb{1}\{f(X_i) \neq Y_i\} =: R_n(f)$$

Theorem 2.15

Suppose that $\mathcal{F}$ is a class of Boolean function with $\mathsf{vc}(\mathcal{F}) < \infty$. Then with probability exceeding $1 - \delta$,

$$R(\widehat{f}_n) - R(f^\star) \leq C\sqrt{\frac{\mathsf{vc}(\mathcal{F}) \log(1/\delta)}{n}}$$

for some universal constant $C > 0$.

Implications:

- For $\mathcal{F} =$ linear classifiers in $\mathbb{R}^d$, the excess risk is $O(\sqrt{d/n})$.